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ORIGINAL PAPER

The interplay between two Euler–Lagrange operators relating to the nonlinear elliptic system $\Sigma[(u,\mathscr{P}),\Omega]$

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Abstract
We establish the existence of multiple whirling solutions to a class of nonlinear elliptic systems in variational form subject to pointwise gradient constraint and pure Dirichlet type boundary conditions. A reduced system for certain $\mathbf{SO}(n)$-valued matrix fields, a description of its solutions via Lie exponentials, a structure theorem for multi-dimensional curl free vector fields and a remarkable explicit relation between two Euler–Lagrange operators of constrained and unconstrained types are the underlying tools and ideas in proving the main result.

Keywords Nonlinear elliptic systems · Incompressible mappings · Euler–Lagrange operators · Multi-dimensional curl operator · Lie exponentials · $\mathbf{SO}(n)$-valued fields · Spherical decomposition

Mathematics Subject Classification 35J57 · 35J62 · 47F10 · 53C22 · 58J70 · 58D19 · 35J50 · 22E30

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ (with $n \geq 2$) be a bounded domain with a $\mathscr{C}^1$ boundary $\partial\Omega$ and consider the variational energy integral

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$$\mathbb{I}[u,\Omega] = \int_\Omega \mathscr{F}(x,u,\nabla u)\,\mathrm{d}x, \tag{1.1}$$

where $\mathscr{F} = \mathscr{F}(x,u,\zeta)$ with $(x,u,\zeta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ is a sufficiently regular real-valued Lagrangian satisfying certain bounds and growth at infinity and the competing maps $u = (u_1,\ldots,u_n)$ are confined to the space of admissible incompressible Sobolev class maps $\mathscr{A}^p_\varphi(\Omega) := \{u \in W^{1,p}(\Omega,\mathbb{R}^n) : \det\nabla u = 1$ a.e. in Ω, $u = \varphi$ on $\partial\Omega\}$ for a fixed choice of exponent $1 \le p < \infty$.

In the above formulation ∇u denotes the gradient matrix of u, an $n \times n$ matrix field in Ω, that here is additionally required to satisfy the hard pointwise incompressibility constraint $\det\nabla u = 1$ in Ω and $\varphi \in \mathscr{C}^1(\partial\Omega;\mathbb{R}^n)$ is a pre-assigned boundary condition. The Euler–Lagrange system associated with the energy integral $\mathbb{I}[u,\Omega]$ over the space of admissible incompressible maps $\mathscr{A}^p_\varphi(\Omega)$ then takes the form (cf., e.g., [1, 3, 4, 6, 19])

$$\Sigma[(u,\mathscr{P}),\Omega] = \begin{cases} \mathscr{L}[u,\mathscr{F}] = \nabla\mathscr{P} & \text{in } \Omega, \\ \det\nabla u = 1 & \text{in } \Omega, \\ u \equiv \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathscr{P} = \mathscr{P}(x)$ is an unknown hydrostatic pressure field (a Lagrange multiplier) corresponding to the pointwise constraint $\det\nabla u = 1$ and the differential operator $\mathscr{L} = \mathscr{L}[u,\mathscr{F}]$ takes the explicit form

$$\mathscr{L}[u,\mathscr{F}] = \frac{1}{2}[\operatorname{cof}\nabla u]^{-1}\{\operatorname{div}[\mathscr{F}_\zeta(x,u,\nabla u)] - \mathscr{F}_u(x,u,\nabla u)\}. \tag{1.3}$$

The divergence operator "div" in the first term on the right acts row-wise on the matrix field $\mathscr{F}_\zeta(x,u,\nabla u)$ whilst $\operatorname{cof}\nabla u$ denotes the cofactor matrix of ∇u. Note that in view of $\det\nabla u = 1$ the cofactor matrix $\operatorname{cof}\nabla u$ is invertible: $\det\operatorname{cof}\nabla u = (\det\nabla u)^{n-1} = 1$ and $[\operatorname{cof}\nabla u]^{-1} = [\nabla u]^t$. Without going into technical details we recall that the system is formally the Euler–Lagrange equation associated with the unconstrained variational energy integral[1]

$$\mathbb{I}^{\mathscr{P}}_{uc}[u,\Omega] = \int_\Omega \mathscr{F}^{\mathscr{P}}_{uc}(x,u,\nabla u)\,\mathrm{d}x = \int_\Omega \{\mathscr{F}(x,u,\nabla u) - 2\mathscr{P}(x)(\det\nabla u - 1)\}\,\mathrm{d}x. \tag{1.4}$$

(Notice that $\mathbb{I}^{\mathscr{P}}_{uc}[u,\Omega] = \mathbb{I}[u,\Omega]$ whenever $u \in \mathscr{A}^p_\varphi(\Omega)$.) Here by a solution to the system (1.2) we mean a pair $(u,\mathscr{P})$ where u is of class $\mathscr{C}^2(\Omega,\mathbb{R}^n) \cap \mathscr{C}(\overline{\Omega},\mathbb{R}^n)$, $\mathscr{P}$ is of class $\mathscr{C}^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$ and the pair satisfy the system (1.2) in the pointwise (classical) sense. If the choice of $\mathscr{P}$ is clear from the context we often abbreviate by saying that u is a solution.

[1] The formal derivation follows the standard route and uses the Piola identity (see, e.g., [3, 4, 27]).

A good motivating source for considering such energies and classes of maps comes from the nonlinear theory of elasticity where the pair (1.1)–(1.2) describe a mathematical model of an incompressible hyperelastic material subject to pure displacement boundary conditions with the resulting extremisers—equivalently critical points or solutions to the associated Euler–Lagrange system—and minimisers serving as the equilibrium states and physically stable displacement fields. (For more on this see [1, 3, 4, 6, 18, 19, 21, 24] and for other motivations see [2, 9, 10, 12, 14, 16, 17, 20, 23, 26, 27] and the references therein.)

Whilst the methods of critical point theory provide a standard and efficient way of establishing the existence of (multiple) solutions to variational problems, due to the complex nature of the incompressibility constraint on the gradient of the competing maps, here, these methods drastically fail and are not applicable. In more technical terms the space $\mathscr{A}^p_\varphi(\Omega)$ is far from being a Hilbert or Banach manifold whilst due to the a priori unknown regularity of the pressure field $\mathscr{P}$, and integrability of the Jacobian determinant $\det \nabla u$, the unconstrained energy integral $\mathbb{I}^{\mathscr{P}}_{uc}$ need not be everywhere well-defined, let alone, being continuously Frechet differentiable.

In this paper, we confine to $\mathscr{F}(x,u,\zeta) = F(r,|u|^2,|\zeta|^2)$ with $F = F(r,s,\xi)$ being a twice continuously differentiable Lagrangian satisfying suitable growth, coercivity and convexity properties (see below for more). Here $r = |x|$, $s = |u|^2 = \langle u,u\rangle$ denotes the 2-norm squared of $u \in \mathbb{R}^n$ and $\xi = |\zeta|^2 = \mathrm{Tr}\,\{\zeta^t\zeta\} = \mathrm{Tr}\,\{\zeta\zeta^t\}$ is the Hilbert-Schmidt norm squared of $\zeta \in \mathbb{R}^{n\times n}$. Thus with this notation in place we have

$$\mathbb{I}[u,\Omega] = \int_\Omega F(|x|,|u|^2,|\nabla u|^2)\,\mathrm{d}x, \tag{1.5}$$

where $|\nabla u|^2 = \mathrm{Tr}\{[\nabla u]^t[\nabla u]\} = \mathrm{Tr}\{[\nabla u][\nabla u]^t\}$, whilst referring to the Euler–Lagrange differential operator $\mathscr{L} = \mathscr{L}[u,\mathscr{F}]$ in (1.3),

$$\mathscr{F}_\zeta(x,u,\nabla u) = 2F_\xi(r,|u|^2,|\nabla u|^2)\nabla u, \qquad \mathscr{F}_u(x,u,\nabla u) = 2F_s(r,|u|^2,|\nabla u|^2)u, \tag{1.6}$$

with F_s and F_ξ denoting the derivatives of the Lagrangian F with respect to the second and third variables, respectively. As a result, abbreviating $\mathscr{L}[u] = \mathscr{L}[u,F]$, the operator becomes

$$\begin{aligned}\mathscr{L}[u] &= [\nabla u]^t\Big\{\mathrm{div}\Big[F_\xi(|x|,|u|^2,|\nabla u|^2)\nabla u\Big] - F_s(|x|,|u|^2,|\nabla u|^2)u\Big\}\\ &= [\nabla u]^t[\nabla u]\nabla F_\xi(|x|,|u|^2,|\nabla u|^2) + F_\xi(|x|,|u|^2,|\nabla u|^2)[\nabla u]^t\Delta u\\ &\quad - F_s(|x|,|u|^2,|\nabla u|^2)[\nabla u]^t u.\end{aligned} \tag{1.7}$$

Further expansion then gives[2]

[2] The identity map $u \equiv x$ is one solution to this system in view of the vector field $\mathscr{L}[u\equiv x] = \nabla[F_\xi] - F_s x$ with $F_\xi = F_\xi(r,r^2,n)$, $F_s = F_s(r,r^2,n)$ being a gradient field in Ω.

$$\begin{aligned}\mathscr{L}[u] = {} & F_{\xi\xi}(|x|,|u|^2,|\nabla u|^2)[\nabla u]^t[\nabla u]\nabla(|\nabla u|^2) + F_{s\xi}(|x|,|u|^2,|\nabla u|^2)[\nabla u]^t[\nabla u]\nabla(|u|^2)\\ & + F_{r\xi}(|x|,|u|^2,|\nabla u|^2)[\nabla u]^t[\nabla u]\nabla|x| + F_{\xi}(|x|,|u|^2,|\nabla u|^2)[\nabla u]^t\Delta u\\ & - F_s(|x|,|u|^2,|\nabla u|^2)[\nabla u]^t u.\end{aligned} \tag{1.8}$$

Our primary task is to establish the existence of multiple solutions to the system $\Sigma[(u,\mathscr{P}),\Omega]$ in (1.2) with $\mathscr{L}=\mathscr{L}[u]$ as given in (1.7). We do so by way of analysing a reduced energy and an associated PDE system for certain $\mathbf{SO}(n)$-valued matrix fields. The solutions u here are in the form of topologically whirling incompressible self-maps of the underlying spatial domain satisfying $|u|=|x|$ and $u|u|^{-1}=\mathbf{Q}[\mathsf{f}(y)]x|x|^{-1}$ (see Sect. 2 for details) whose analytic and geometric features are intimately linked to those of the Lie group $\mathbf{SO}(n)$ and its Lie algebra of skew-symmetric matrices $\mathfrak{so}(\mathfrak{n})$. Here $\Omega\subset\mathbb{R}^n$ is taken to be a bounded open annulus whilst $\mathbf{Q}=\mathbf{Q}[\mathsf{f}]$ is an $\mathbf{SO}(n)$-valued field depending on a vector map $\mathsf{f}=(f_1,\ldots,f_d)$ that ultimately relates to the spatial variables $x=(x_1,\ldots,x_n)$ through a vector of 2-plane radial variables $y=(y_1,\ldots,y_N)$ lying in a semi-annular region $\mathbb{A}_n\subset\mathbb{R}^N$. The pair $N\geq 1$ and $d\geq 1$ are suitable integers relating to $n\geq 2$ (see Sect. 3). The PDE for the vector map $\mathsf{f}=(f_1,\ldots,f_d)$ is then the strictly elliptic unconstrained system (with $1\leq i\leq d$)

$$\mathbf{RS}[\mathsf{f},\mathbb{A}_n] = \begin{cases} \operatorname{div}[\mathsf{A}_i(y,\nabla\mathsf{f})\nabla f_i]=0 & \text{in } \mathbb{A}_n,\\ \mathsf{f}\equiv 0 & \text{on } (\partial\mathbb{A}_n)_a,\\ \mathsf{f}\equiv 2\mathsf{m}\pi & \text{on } (\partial\mathbb{A}_n)_b,\\ \mathsf{A}_i(y,\nabla\mathsf{f})\partial_\nu f_i = 0 & \text{on } \Gamma_n,\end{cases} \tag{1.9}$$

where $\nabla\mathsf{f}=[\partial_j f_i : 1\leq i\leq d, 1\leq j\leq N]$, $\mathsf{m}=(m_1,\ldots,m_d)\in\mathbb{Z}^d$, $\mathscr{J}(y)=y_1\ldots y_d$ is a suitable Jacobian, and

$$\mathsf{A}_i(y,\nabla\mathsf{f}) = F_\xi\left(||y||,||y||^2,n+\sum_{j=1}^d y_j^2|\nabla f_j|^2\right)y_i^2\mathscr{J}(y), \qquad ||y||^2=\sum_{j=1}^N y_j^2, \tag{1.10}$$

(see Sect. 4 for details). A thorough analysis then leads to a remarkable relationship between the two systems and their corresponding differential operators given by

$$\begin{aligned}\mathscr{L}[u] = {} & \nabla F_\xi - F_s x - \sum_{\ell=1}^N F_\xi \partial_\ell \mathbf{Q}^t\partial_\ell\mathbf{Q}x\\ & + \frac{1}{\mathscr{J}(y)}\left[\mathbf{I}_n + \sum_{\ell=1}^N \nabla y_\ell\otimes\mathbf{Q}^t\partial_\ell\mathbf{Q}x\right]\left\{\sum_{i=1}^d\frac{1}{y_i^2}\operatorname{div}[\mathsf{A}_i(y,\nabla\mathsf{f})\nabla f_i][w^i]^\perp\right\}.\end{aligned} \tag{1.11}$$

Here, the vectors w^i for $1\leq i\leq d$ are n-vectors introduced in Sect. 4 with $[w^i]^\perp$ their orthogonal counterparts. Thus, in particular, if $\mathsf{f}=(f_1,\ldots,f_d)$ is a solution to the system (1.9) then the above leads to $\mathscr{L}[u]=\nabla F_\xi - F_s x - \sum_{\ell=1}^N F_\xi\partial_\ell\mathbf{Q}^t\partial_\ell\mathbf{Q}x$.

As a result the task of resolving the PDE $\mathscr{L}[u] = \nabla\mathscr{P}$ shifts to verifying whether and when the vector field on the right-hand side is a gradient. This analysis will be carried out by studying certain classes of curl free vector fields and an associated structure theorem paving the way for the main existence and multiplicity result formulated and proved in the final two sections of the paper.

Assumptions on F. Let us end by describing the regularity and convexity assumptions imposed on the Lagrangian $F = F(r, s, \xi)$. First we assume throughout that $F \in \mathscr{C}^2(U)$ where $U = [a, b] \times]0, \infty[\times]0, \infty[\subset \mathbb{R}^3$. Next we assume F to be bounded from below on U with $F_\xi > 0$, $F_{\xi\xi} \geq 0$. Moreover, we assume that for every compact set $K \subset]0, \infty[$ there exist real constants c_0, c_1, c_2 depending on K such that, for $p > 1$,

$$F_\xi(r, s, \zeta^2)|\zeta| \leq c_2|\zeta|^{p-1} \qquad \forall\,(r, s, \zeta^2) \in U : \ s \in K, \tag{1.12}$$

$$c_0 + c_1|\zeta|^p \leq F(r, s, \zeta^2) \leq c_2|\zeta|^p \qquad \forall\,(r, s, \zeta^2) \in U : \ s \in K. \tag{1.13}$$

Finally, the twice continuously differentiable function $\zeta \mapsto F(r, r^2, n + r^2\zeta^2)$ is assumed to be uniformly convex in ζ for all $a \leq r \leq b$ and $\zeta \in \mathbb{R}$.

2 Radial and Spherical Decompositions $\mathscr{R}_u$,$\mathscr{S}_u$ and a Reformulation of $\mathscr{L}[u]$ in $\Sigma[(u,\mathscr{P}),\Omega]$

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain and let $u \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy $|u| > 0$ a.e. in Ω. We decompose u into a radial part $\mathscr{R}_u$ and a spherical part $\mathscr{S}_u$, respectively, by setting $\mathscr{R}_u = |u|$ and $\mathscr{S}_u = u|u|^{-1}$. As u is (weakly) differentiable basic calculation gives

$$\nabla\mathscr{R}_u = |u|^{-1}[\nabla u]^t u, \qquad \nabla\mathscr{S}_u = |u|^{-1}\Big(\mathbf{I}_n - u|u|^{-1} \otimes u|u|^{-1}\Big)\nabla u, \tag{2.1}$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix and as before $[\nabla u]^t$ is the transpose of $[\nabla u]$. We also introduce a pair of matrix-fields associated with u and intertwined with the PDE:

$$\mathbf{X}[u] := [\nabla u]^t[\nabla u] - \mathbf{I}_n, \qquad \mathbf{Y}[u] := [\nabla u][\nabla u]^t - \mathbf{I}_n. \tag{2.2}$$

These in a way measure the closeness of the gradient field ∇u to the orthogonal group $\mathbf{O}(n)$ and hence the deformation u to a rigid motion by Liouville's theorem (evidently $\nabla u \in \mathbf{O}(n) \iff \mathbf{X}[u] \equiv 0 \iff \mathbf{Y}[u] \equiv 0$). Let us proceed by listing some of the main quantities associated with u in terms of its radial and spherical parts.

Lemma 1 *With the notation on $\mathscr{R}_u = |u|$ and $\mathscr{S}_u = u|u|^{-1}$ as above the following identities hold*:

(i) $\nabla u = \mathscr{R}_u\nabla\mathscr{S}_u + \mathscr{S}_u \otimes \nabla\mathscr{R}_u$,

(ii) $[\nabla u]^t u = \mathscr{R}_u(\mathscr{R}_u[\nabla\mathscr{S}_u]^t + \nabla\mathscr{R}_u \otimes \mathscr{S}_u)\mathscr{S}_u = \mathscr{R}_u\nabla\mathscr{R}_u$,

(iii) $\mathbf{X}[u] = \mathscr{R}_u^2[\nabla\mathscr{S}_u]^t[\nabla\mathscr{S}_u] + \nabla\mathscr{R}_u \otimes \nabla\mathscr{R}_u - \mathbf{I}_n$,

(iv) $\mathbf{Y}[u] = \mathscr{R}_u^2[\nabla \mathscr{S}_u][\nabla \mathscr{S}_u]^t + \mathscr{R}_u \nabla \mathscr{S}_u \nabla \mathscr{R}_u \otimes \mathscr{S}_u + \mathscr{R}_u \mathscr{S}_u \otimes \nabla \mathscr{S}_u \nabla \mathscr{R}_u + |\nabla \mathscr{R}_u|^2 \mathscr{S}_u \otimes \mathscr{S}_u - \mathbf{I}_n,$

(v) $|\nabla u|^2 = \mathrm{Tr}\{[\nabla u]^t[\nabla u]\} = \mathrm{Tr}\{[\nabla u][\nabla u]^t\} = \mathscr{R}_u^2|\nabla \mathscr{S}_u|^2 + |\nabla \mathscr{R}_u|^2.$

If, additionally, u is twice differentiable, then

(vi) $\Delta u = \mathscr{R}_u \Delta \mathscr{S}_u + 2\nabla \mathscr{S}_u \nabla \mathscr{R}_u + \Delta \mathscr{R}_u \mathscr{S}_u,$

(vii) $[\nabla u]^t \Delta u = \mathscr{R}_u^2[\nabla \mathscr{S}_u]^t \Delta \mathscr{S}_u + \Delta \mathscr{R}_u \nabla \mathscr{R}_u + \mathscr{R}_u(2[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u] + \langle \mathscr{S}_u, \Delta \mathscr{S}_u \rangle \mathbf{I}_n)\nabla \mathscr{R}_u.$

Proof These identities are all consequences of the definition and follow by direct differentiation upon noting $|\mathscr{S}_u|^2 = 1$, $[\nabla \mathscr{S}_u]^t \mathscr{S}_u = 0$. The details are left to the reader.

Proposition 1 *The partial differential action $\mathscr{L}[u]$ in (1.8) can be formulated in the radial and spherical parts $\mathscr{R}_u$ and $\mathscr{S}_u$ as:*

$$\mathscr{L}[u] = F_{\xi\xi}\mathsf{A}[u] + F_{s\xi}\mathsf{B}[u] + F_{r\xi}\mathsf{C}[u] + F_{\xi}\mathsf{D}[u] + F_s\mathsf{E}[u], \tag{2.3}$$

where the arguments of F and all subsequent derivatives are evaluated at the vector point $(|x|, |u|^2, |\nabla u|^2) = (r, \mathscr{R}_u^2, \mathscr{R}_u^2|\nabla \mathscr{S}_u|^2 + |\nabla \mathscr{R}_u|^2)$ whilst

$$\begin{aligned} \mathsf{A}[u] = \mathsf{A}[\mathscr{R}_u, \mathscr{S}_u] &:= [\nabla u]^t[\nabla u]\nabla(|\nabla u|^2) \\ &= \left[\mathscr{R}_u^2[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u] + \nabla \mathscr{R}_u \otimes \nabla \mathscr{R}_u\right]\nabla\left[\mathscr{R}_u^2|\nabla \mathscr{S}_u|^2 + |\nabla \mathscr{R}_u|^2\right]; \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mathsf{B}[u] = \mathsf{B}[\mathscr{R}_u, \mathscr{S}_u] &:= [\nabla u]^t[\nabla u]\nabla(|u|^2) \\ &= 2\mathscr{R}_u\left[\mathscr{R}_u^2[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u]\nabla \mathscr{R}_u + |\nabla \mathscr{R}_u|^2 \nabla \mathscr{R}_u\right]; \end{aligned} \tag{2.5}$$

$$\mathsf{C}[u] = \mathsf{C}[\mathscr{R}_u, \mathscr{S}_u] := [\nabla u]^t[\nabla u]\nabla|x| = \left[\mathscr{R}_u^2[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u] + \nabla \mathscr{R}_u \otimes \nabla \mathscr{R}_u\right]x|x|^{-1}; \tag{2.6}$$

$$\begin{aligned} \mathsf{D}[u] = \mathsf{D}[\mathscr{R}_u, \mathscr{S}_u] &:= [\nabla u]^t \Delta u = \mathscr{R}_u^2[\nabla \mathscr{S}_u]^t \Delta \mathscr{S}_u + 2\mathscr{R}_u[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u]\nabla \mathscr{R}_u \\ &\quad + \mathscr{R}_u\langle \mathscr{S}_u, \Delta \mathscr{S}_u\rangle \nabla \mathscr{R}_u + \Delta \mathscr{R}_u \nabla \mathscr{R}_u; \end{aligned} \tag{2.7}$$

and finally $\mathsf{E}[u] = \mathsf{E}[\mathscr{R}_u, \mathscr{S}_u] := -[\nabla u]^t u = -\mathscr{R}_u \nabla \mathscr{R}_u.$

Proof We invoke the definition of $\mathscr{L}[u]$ in (1.8) and the identities gathered in Lemma 1. For the vector field $\mathsf{B}[u]$ we note that $\nabla(|u|^2) = 2\mathscr{R}_u \nabla \mathscr{R}_u$. Pre-multiplying by $[\nabla u]^t[\nabla u]$ gives (2.5). The identity (2.6) then follows upon noting that $\nabla|x| = x/|x|$. The descriptions of the vector fields $\mathsf{A}[u]$, $\mathsf{D}[u]$ and $\mathsf{E}[u]$ are taken directly from identities (ii)–(vii) in Lemma 1.

3 Incompressible and topological twists of annuli

In this section, we specialise to the situation where $\Omega \subset \mathbb{R}^n$ is a symmetric open bounded annulus, for definiteness, $\Omega = \mathbb{X}_n = \mathbb{X}_n[a,b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ with $b > a > 0$ and $\varphi \equiv \mathbf{I}_{\overline{\Omega}}$ (identity map). The choice of φ is to avoid unnecessary technicalities without losing too much generality, whilst the choice of domain geometry is prompted by applications to multiplicity results we have in mind for later on (compare with, e.g., [13, 22] as well as [1, 6, 7, 11, 24, 28]).

For a self-map $u \in \mathscr{C}(\overline{\mathbb{X}}_n, \overline{\mathbb{X}}_n)$, recall the notation introduced in the previous section, specifically, the radial part $\mathscr{R}_u$ and the spherical part $\mathscr{S}_u$ given by $\mathscr{R}_u := |u| \in \mathscr{C}(\overline{\mathbb{X}}_n, [a,b])$ and $\mathscr{S}_u := u|u|^{-1} \in \mathscr{C}(\overline{\mathbb{X}}_n, \mathbb{S}^{n-1})$, respectively. If $u \equiv x$ on $\partial\mathbb{X}_n$ then $\mathscr{R}_u \equiv a$ and $\mathscr{R}_u \equiv b$ on the inner and outer components of $\partial\mathbb{X}_n$, respectively, whilst $\mathscr{S}_u \equiv \theta$ on $\partial\mathbb{X}_n$. Furthermore, due to the cartesian product structure of $\mathbb{X}_n$, the spherical part $\mathscr{S}_u$ can be seen, with a slight abuse of notation, to verify $\mathscr{S}_u \in \mathscr{C}([a,b]; \mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = 1))$ with $\mathscr{S}_u|_{r=a} = \mathbf{I}_{\mathbb{S}^{n-1}} = \mathscr{S}_u|_{r=b}$ where $\mathbf{I}_{\mathbb{S}^{n-1}}$ denotes the identity map of the unit sphere. Here we write $\mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = d)$ $(d \in \mathbb{Z})$ for the connected component of the mapping space $\mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})$ consisting of maps with Hopf degree d. As a result $\mathscr{S}_u$ represents an element of the fundamental group $\pi_1[\mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = 1)] \cong \pi_1[\mathbf{SO}(n)]$ (for more on this see [8, 24, 25, 29, 30]). Conversely any map $\mathscr{S} = \mathscr{S}(r)$ in $\mathscr{C}([a,b]; \mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = 1))$ satisfying $\mathscr{S}(a) = \mathscr{S}(b) = \mathbf{I}_{\mathbb{S}^{n-1}}$ gives rise to a self-map $u \in \mathscr{C}(\overline{\mathbb{X}}_n, \overline{\mathbb{X}}_n)$ with $u \equiv x$ on $\partial\mathbb{X}_n$ through the recipe $\mathscr{R}_u(x) = f(|x|)$ and $\mathscr{S}_u \equiv \mathscr{S}$, i.e., $u : (r, \theta) \mapsto (f(r), \mathscr{S}(r)[\theta])$. Here $f \in \mathscr{C}([a,b],[a,b])$ is any function satisfying $f(a) = a$ and $f(b) = b$ (e.g., $f(r) \equiv r$). In what follows we look at particular classes of self-maps u whose spherical parts $\mathscr{S}_u$ result from an $\mathbf{SO}(n)$-valued matrix field $\mathbf{Q}$ as described in (a)–(b) below.

(a) **Twists** $u \in \mathscr{C}(\overline{\mathbb{X}}_n, \overline{\mathbb{X}}_n)$. By a generalised twist or simply a *twist* we understand a self-map u whose radial and spherical parts are given by

$$\mathscr{R}_u(x) = |x|, \quad \mathscr{S}_u(x) = \mathbf{Q}(|x|)x|x|^{-1}, \quad x \in \overline{\mathbb{X}}_n. \tag{3.1}$$

Here the curve $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n))$ is referred to as the twist path associated with u. To ensure $u \equiv x$ on $\partial\Omega = \partial\mathbb{X}_n$ we set $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix. In this event the twist path is a closed curve in $\mathbf{SO}(n)$ based at $\mathbf{I}_n$ thus representing an element of $\pi_1[\mathbf{SO}(n)] \cong \mathbb{Z}_2$ $(n \geq 3)$ and $\cong \mathbb{Z}$ $(n = 2)$. Here we refer to $\mathbf{Q} = \mathbf{Q}(r)$ as the twist *loop* associated to u. Now subject to the differentiability of the twist path $\mathbf{Q}$ (hereafter we write $\dot{\mathbf{Q}} = \mathrm{d}\mathbf{Q}/\mathrm{d}r$, $\ddot{\mathbf{Q}} = \mathrm{d}^2\mathbf{Q}/\mathrm{d}r^2$) we have

$$\nabla\mathscr{R}_u = x|x|^{-1}, \qquad \nabla\mathscr{S}_u = |x|^{-1}[\mathbf{Q} + (|x|\dot{\mathbf{Q}} - \mathbf{Q})x|x|^{-1} \otimes x|x|^{-1}], \tag{3.2}$$

and so a direct calculation (see below) leads to $\det \nabla u = 1$. If u is twice differentiable then by taking second derivatives it can be easily seen that $\Delta\mathscr{R}_u = (n-1)/|x|$ and $\Delta\mathscr{S}_u = [-(n-1)|x|^{-2}\mathbf{Q}x + (n-1)|x|^{-1}\dot{\mathbf{Q}}x + \ddot{\mathbf{Q}}x]/|x|$.

Lemma 2 *Let u be a twist associated with $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^1(]a,b[, \mathbf{SO}(n))$. Then the following hold:*

(i) $\nabla u = \mathbf{Q} + |x|^{-1}\dot{\boldsymbol{Q}}x \otimes x$,

(ii) $\det \nabla u = \det[\mathbf{Q} + |x|^{-1}\dot{\boldsymbol{Q}}x \otimes x] = 1$,

(iii) $[\nabla u]^t u = [\mathbf{Q}^t + |x|^{-1} x \otimes \dot{\boldsymbol{Q}}x]\mathbf{Q}x = x$,

(iv) $\mathbf{X}[u] = |x|^{-1}[\mathbf{Q}^t\dot{\boldsymbol{Q}}x \otimes x + x \otimes \mathbf{Q}^t\dot{\boldsymbol{Q}}x] + |x|^{-2}|\dot{\boldsymbol{Q}}x|^2 x \otimes x$,

(v) $\mathbf{Y}[u] = |x|^{-1}[\dot{\boldsymbol{Q}}x \otimes \mathbf{Q}x + \mathbf{Q}x \otimes \dot{\boldsymbol{Q}}x] + \dot{\boldsymbol{Q}}x \otimes \dot{\boldsymbol{Q}}x$,

(vi) $|\nabla u|^2 = n + |\dot{\boldsymbol{Q}}x|^2$.

If, moreover, the matrix field $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathbf{SO}(n))$ then we have

(vii) $\Delta u = (n+1)\dot{\boldsymbol{Q}}x|x|^{-1} + \ddot{\boldsymbol{Q}}x$,

(viii) $[\nabla u]^t \Delta u = (n+1)\left[|x|\mathbf{Q}^t\dot{\boldsymbol{Q}}x + |\dot{\boldsymbol{Q}}x|^2 x\right]|x|^{-2} + \mathbf{Q}^t\ddot{\boldsymbol{Q}}x + \langle \dot{\boldsymbol{Q}}x, \ddot{\boldsymbol{Q}}x\rangle x|x|^{-1}$.

Proof We make use of Lemma 1. First, using (3.2), $\nabla u = \mathscr{R}_u \nabla \mathscr{S}_u + \mathscr{S}_u \otimes \nabla \mathscr{R}_u = \mathbf{Q} - |x|^{-2}\mathbf{Q}x \otimes x + |x|^{-1}\dot{\boldsymbol{Q}}x \otimes x + |x|^{-2}\mathbf{Q}x \otimes x$ giving (i). For (ii) using the skew-symmetry of $\mathbf{Q}^t\dot{\boldsymbol{Q}}$ we have $\det[\mathbf{Q} + |x|^{-1}\dot{\boldsymbol{Q}}x \otimes x] = \det[\mathbf{I}_n + |x|^{-1}\mathbf{Q}^t\dot{\boldsymbol{Q}}x \otimes x] = 1 + |x|^{-1}\langle \mathbf{Q}^t\dot{\boldsymbol{Q}}x, x\rangle = 1$. Next $[\nabla u]^t u = \mathscr{R}_u \nabla \mathscr{R}_u = x$ giving (iii). For $\mathbf{X}[u]$ we use the relation $[\nabla u]^t[\nabla u] = \nabla \mathscr{R}_u \otimes \nabla \mathscr{R}_u + \mathscr{R}_u^2[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u]$ together with

$$[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u] = |x|^{-2}\left[\mathbf{I}_n + |x|^{-1}(\mathbf{Q}^t\dot{\boldsymbol{Q}}x \otimes x + x \otimes \mathbf{Q}^t\dot{\boldsymbol{Q}}x) + |x|^{-2}(|\dot{\boldsymbol{Q}}x|^2 - 1)x \otimes x\right],$$

giving (iv). For $\mathbf{Y}[u]$ we use $[\nabla u][\nabla u]^t = \mathscr{R}_u^2[\nabla \mathscr{S}_u][\nabla \mathscr{S}_u]^t + \mathscr{R}_u(\nabla \mathscr{S}_u \nabla \mathscr{R}_u \otimes \mathscr{S}_u + \mathscr{S}_u \otimes \nabla \mathscr{S}_u \nabla \mathscr{R}_u) + |\nabla \mathscr{R}_u|^2 \mathscr{S}_u \otimes \mathscr{S}_u$ together with $\nabla \mathscr{S}_u \nabla \mathscr{R}_u = |x|^{-1}\dot{\boldsymbol{Q}}x$ and

$$[\nabla \mathscr{S}_u][\nabla \mathscr{S}_u]^t = |x|^{-2}\left[\mathbf{I}_n - |x|^{-2}\mathbf{Q}x \otimes \mathbf{Q}x + \dot{\boldsymbol{Q}}x \otimes \dot{\boldsymbol{Q}}x\right],$$

giving (v). Next (vi) results from taking the trace of either of $\mathbf{X}[u]$ or $\mathbf{Y}[u]$. Now moving to the next part, for the Laplacian we use $\Delta u = \mathscr{R}_u \Delta \mathscr{S}_u + 2\nabla \mathscr{S}_u \nabla \mathscr{R}_u + \Delta \mathscr{R}_u \mathscr{S}_u$ along with the earlier calculation of the constituting terms. The final identity can be pieced together using ingredients already gathered in the earlier part of the lemma.

Proposition 2 *Let u be a twist with $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathbf{SO}(n))$. Then*

$$\begin{aligned}\mathscr{L}[u] = {} & 2|x|^{-1}F_{\xi\xi}[|x|\langle \ddot{\boldsymbol{Q}}x, \dot{\boldsymbol{Q}}x\rangle \mathbf{Q}^t\dot{\boldsymbol{Q}}x + |\dot{\boldsymbol{Q}}x|^2\mathbf{Q}^t\dot{\boldsymbol{Q}}x + |\dot{\boldsymbol{Q}}x|^2\langle \ddot{\boldsymbol{Q}}x, \dot{\boldsymbol{Q}}x\rangle x + |\dot{\boldsymbol{Q}}x|^4 x|x|^{-1}] \\ & + \left[2F_{s\xi} + |x|^{-1}F_{r\xi}\right]\left[|x|\mathbf{Q}^t\dot{\boldsymbol{Q}}x + |\dot{\boldsymbol{Q}}x|^2 x\right] + \nabla F_\xi \\ & + |x|^{-1}F_\xi[(n+1)\mathbf{Q}^t\dot{\boldsymbol{Q}}x + (n+1)|x|^{-1}|\dot{\boldsymbol{Q}}x|^2 x + |x|\mathbf{Q}^t\ddot{\boldsymbol{Q}}x + \langle \dot{\boldsymbol{Q}}x, \ddot{\boldsymbol{Q}}x\rangle x] - F_s x.\end{aligned} \tag{3.3}$$

Here $\mathscr{L}$ is the differential operator in (1.8) and the arguments of $F = F(r, s, \xi)$ and all subsequent derivatives are $(r, s, \xi) = (r, r^2, n + |\dot{\mathbf{Q}}x|^2)$.[3]

Proof We justify the statement using Proposition 1 and calculating the associated coefficients. First for $\mathsf{A}[u]$, noting $|\nabla \mathscr{R}_u|^2 = 1$, $|\nabla \mathscr{S}_u|^2 = |x|^{-2}[n - 1 + |\dot{\mathbf{Q}}x|^2]$, we have

$$\begin{aligned}\mathsf{A}[u] &= \left[\mathscr{R}_u^2[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u] + \nabla \mathscr{R}_u \otimes \nabla \mathscr{R}_u\right]\nabla\left[\mathscr{R}_u^2|\nabla \mathscr{S}_u|^2 + |\nabla \mathscr{R}_u|^2\right] \\ &= \left[\mathbf{I}_n + |x|^{-1}(\mathbf{Q}^t\dot{\mathbf{Q}}x \otimes x + x \otimes \mathbf{Q}^t\dot{\mathbf{Q}}x) + |x|^{-2}|\dot{\mathbf{Q}}x|^2 x \otimes x\right]\nabla\left[n + |\dot{\mathbf{Q}}x|^2\right]\end{aligned}$$

where then $\nabla[n + |\dot{\mathbf{Q}}x|^2] = \nabla|\dot{\mathbf{Q}}x|^2 = 2\langle \ddot{\mathbf{Q}}x, \dot{\mathbf{Q}}x\rangle x|x|^{-1} + 2\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}x$. Next, for $\mathsf{B}[u]$, $\mathsf{C}[u]$ we have $\mathsf{B}[u] = 2\mathscr{R}_u[\mathscr{R}_u^2[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u]\nabla \mathscr{R}_u + |\nabla \mathscr{R}_u|^2\nabla \mathscr{R}_u] = 2[x + |x|\mathbf{Q}^t\dot{\mathbf{Q}}x + |\dot{\mathbf{Q}}x|^2 x]$ and $\mathsf{C}[u] = [\mathscr{R}_u^2[\nabla \mathscr{S}_u]^t[\nabla \mathscr{S}_u] + \nabla \mathscr{R}_u \otimes \nabla \mathscr{R}_u]x|x|^{-1} = [x + |x|\mathbf{Q}^t\dot{\mathbf{Q}}x + |\dot{\mathbf{Q}}x|^2 x]/|x|$. The remaining coefficients $\mathsf{D}[u]$ and $\mathsf{E}[u]$ can be taken from (iii) and (viii) in Lemma 2. The conclusion now follows upon noting $\nabla F_\xi = F_{\xi\xi}\nabla|\dot{\mathbf{Q}}x|^2 + (2F_{s\xi} + |x|^{-1}F_{r\xi})x$.

(b) **Whirls** $u \in \mathscr{C}(\overline{\mathbb{X}}_n, \overline{\mathbb{X}}_n)$. By a whirl map or a whirl for simplicity we understand a self-map u whose radial and spherical parts have the forms

$$\mathscr{R}_u(x) = |x|, \quad \mathscr{S}_u(x) = \mathbf{Q}(y_1, \ldots, y_N)x|x|^{-1}, \quad x \in \overline{\mathbb{X}}_n. \tag{3.4}$$

Here, we denote by $y = y(x)$ the vector of 2-plane radial variables $(y_1, \ldots, y_N)$, defined, depending on the dimension $n \geq 2$ being even or odd, as follows: If $n = 2N$ we set $y_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2}$ with $1 \leq j \leq N$. If $n = 2N - 1$ we set $y_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2}$ with $1 \leq j \leq N - 1$ and $y_N = x_n$. In the first case set $d = N$ and in the second case set $d = N - 1$. It is now seen that for $x \in \overline{\mathbb{X}}_n$ the vector $y = y(x)$ lies in the semi-annular domain $\overline{\mathbb{A}}_n \subset \mathbb{R}^N$ where $\mathbb{A}_n = \{y \in \mathbb{R}_+^d : a < \|y\| < b\}$ when $n = 2N$ and $\mathbb{A}_n = \{y \in \mathbb{R}_+^d \times \mathbb{R} : a < \|y\| < b\}$ when $n = 2N - 1$ with $\|y\| = (y_1^2 + \cdots + y_N^2)^{1/2}$. We write $\partial\mathbb{A}_n = (\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b \cup \Gamma_n$ where the three disjoint segments of $\partial\mathbb{A}_n$ are defined as: $\Gamma_n = \partial\mathbb{A}_n \setminus [(\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b]$ (the flat part), $(\partial\mathbb{A}_n)_a = \{y \in \partial\mathbb{A}_n : \|y\| = a\}$ and $(\partial\mathbb{A}_n)_b = \{y \in \partial\mathbb{A}_n : \|y\| = b\}$. Note that $x \in (\partial\mathbb{X}_n)_a = \{|x| = a\} \iff y(x) \in (\partial\mathbb{A}_n)_a$, $x \in (\partial\mathbb{X}_n)_b = \{|x| = b\} \iff y(x) \in (\partial\mathbb{A}_n)_b$ whilst the flat part Γ_n does not correspond to any part of $\partial\mathbb{X}_n$.

With this notation in place let us now give a more explicit description of the $\mathbf{SO}(n)$-valued matrix field $\mathbf{Q} = \mathbf{Q}(y_1, \ldots, y_N)$ defining the spherical part $\mathscr{S}_u$.[4] Let us denote by $\mathbf{R}[\alpha]$ the usual $\mathbf{SO}(2)$ matrix of rotation by angle α, specifically,

[3] For an extensive study of solvability and multiple solutions to the system $\Sigma[(u, \mathscr{P}), \mathbb{X}_n]$ in (1.2) in the form of twists the interested reader is referred to our forthcoming paper (see also [18, 19, 21]).

[4] Despite apparent similarities these two classes of maps are different in that in the first case (twists) the dependence of the twist path is on the radial variable $r = |x|$ only with no restriction on its range whereas in the second case (whirls) the dependence is on the 2-plane radial variables $y = (y_1, \ldots, y_N)$ with the

$$\mathbf{R}[\alpha] = \exp\{\alpha \mathbf{J}\} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad \mathbf{J} = \sqrt{-\mathbf{I}_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.5}$$

Here and below we write $X \mapsto \exp\{X\}$ for the Lie exponential map of $\mathbf{SO}(n)$, whose domain is the Lie algebra $\mathfrak{so}(\mathfrak{n})$ of $n \times n$ skew-symmetric real matrices. Consideration of symmetry (see [16, 17]) leads to the matrix field $\mathbf{Q} = \mathbf{Q}[\mathsf{f}](y)$ taking values on a maximal torus $\mathbb{T}$ of $\mathbf{SO}(n)$. Thus, here, the form $\mathbf{Q} = \mathrm{diag}(\mathbf{R}[f_1(y)], \ldots, \mathbf{R}[f_d(y)])$, that is, $\mathbf{Q} = \exp\{\mathrm{diag}(f_1(y)\mathbf{J}, \ldots, f_d(y)\mathbf{J})\}$, or more explicitly, the block diagonal form,

$$\mathbf{Q}[\mathsf{f}](y) = \begin{pmatrix} \mathbf{R}[f_1(y)] & 0 & \ldots & 0 & 0 \\ 0 & \mathbf{R}[f_2(y)] & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbf{R}[f_{d-1}(y)] & 0 \\ 0 & 0 & \ldots & 0 & \mathbf{R}[f_d(y)] \end{pmatrix}, \tag{3.6}$$

for when the dimension $n = 2d$ is even and $\mathbf{Q} = \mathrm{diag}(\mathbf{R}[f_1(y)], \ldots, \mathbf{R}[f_d(y)], 1)$, that is, $\mathbf{Q} = \exp\{\mathrm{diag}(f_1(y)\mathbf{J}, \ldots, f_d(y)\mathbf{J}, 0)\}$, or again more explicitly, the block diagonal form

$$\mathbf{Q}[\mathsf{f}](y) = \begin{pmatrix} \mathbf{R}[f_1(y)] & 0 & \ldots & 0 & 0 & 0 \\ 0 & \mathbf{R}[f_2(y)] & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbf{R}[f_{d-1}(y)] & 0 & 0 \\ 0 & 0 & \ldots & 0 & \mathbf{R}[f_d(y)] & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}, \tag{3.7}$$

for when $n = 2d + 1$ is odd. The functions $f_\ell \in \mathscr{C}(\overline{\mathbb{A}}_n)$ with $1 \le \ell \le d$ are set to satisfy the boundary condition $f_\ell \equiv 0$ on $(\partial \mathbb{A}_n)_a$ and $f_\ell \equiv 2m_\ell \pi$ on $(\partial \mathbb{A}_n)_b$ for suitable $m_\ell \in \mathbb{Z}$. This prompts $\mathbf{Q} \equiv \mathbf{I}_n$ and subsequently $u \equiv x$ on $\partial \mathbb{X}_n$. For obvious reasons we call $f_\ell = f_\ell(y_1, \ldots, y_N)$ $(1 \le \ell \le d)$ the angle of rotation functions and we denote by f the vector $\mathsf{f} = (f_1, \ldots, f_d)$. Now subject to a differentiability assumption on $\mathbf{Q}$ we can write

$$\nabla \mathscr{R}_u = x|x|^{-1}, \qquad \nabla \mathscr{S}_u = |x|^{-1}\left[(\mathbf{Q} - \mathbf{Q}x|x|^{-1} \otimes x|x|^{-1}) + \sum_{\ell=1}^{N} \partial_\ell \mathbf{Q}x \otimes \nabla y_\ell \right] \tag{3.8}$$

(with $\partial_\ell \mathbf{Q} = \partial \mathbf{Q}/\partial y_\ell$) and again after direct but a little more involved calculation (see Lemma 3 below) it follows that $\det \nabla u = 1$. Thus hereafter by a whirl we understand a self-map u as in (3.4) where the matrix field Q in $\mathscr{S}_u$ has either form (3.6) or (3.7).

Footnote 4 continued
range restricted to a maximal torus. Thus whirls have considerably less symmetries than twists (see [15–17]).

Lemma 3 *Let u be a whirl associated with $\mathbf{Q} \in \mathscr{C}(\overline{\mathbb{A}}_n, \mathbf{SO}(n)) \cap \mathscr{C}^1(\mathbb{A}_n, \mathbf{SO}(n))$. Then the following hold*:

(i) $\nabla u = \mathbf{Q} + \sum_{\ell=1}^N \partial_\ell \mathbf{Q}x \otimes \nabla y_\ell$,

(ii) $\det \nabla u = \det\big[\mathbf{Q} + \sum_{\ell=1}^N \partial_\ell \mathbf{Q}x \otimes \nabla y_\ell\big] = 1$,

(iii) $[\nabla u]^t u = \big[\mathbf{Q}^t + \sum_{\ell=1}^N \nabla y_\ell \otimes \partial_\ell \mathbf{Q}x\big]\mathbf{Q}x = x$,

(iv) $\mathbf{X}[u] = \sum_{\ell=1}^N [\mathbf{Q}^t\partial_\ell\mathbf{Q}x \otimes \nabla y_\ell + \nabla y_\ell \otimes \mathbf{Q}^t\partial_\ell\mathbf{Q}x] + \sum_{\ell=1}^N \sum_{k=1}^N \langle \partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}x\rangle \nabla y_\ell \otimes \nabla y_k$,

(v) $\mathbf{Y}[u] = \sum_{\ell=1}^N [\mathbf{Q}\nabla y_\ell \otimes \partial_\ell\mathbf{Q}x + \partial_\ell\mathbf{Q}x \otimes \mathbf{Q}\nabla y_\ell + \partial_\ell\mathbf{Q}x \otimes \partial_\ell\mathbf{Q}x]$,

(vi) $|\nabla u|^2 = n + \sum_{\ell=1}^N \Big[2\langle \mathbf{Q}^t\partial_\ell\mathbf{Q}x, \nabla y_\ell\rangle + |\partial_\ell\mathbf{Q}x|^2\Big] = n + \sum_{\ell=1}^N |\partial_\ell\mathbf{Q}x|^2$.

If, moreover, the matrix field $\mathbf{Q} \in \mathscr{C}(\overline{\mathbb{A}}_n, \mathbf{SO}(n)) \cap \mathscr{C}^2(\mathbb{A}_n, \mathbf{SO}(n))$ then we have

(vii) $\Delta u = \sum_{\ell=1}^N \big[\partial_\ell^2\mathbf{Q}x + \Delta y_\ell \partial_\ell\mathbf{Q}x + 2\partial_\ell\mathbf{Q}\nabla y_\ell\big]$,

(viii) $[\nabla u]^t \Delta u = \sum_{\ell=1}^N \big[\mathbf{Q}^t\partial_\ell^2\mathbf{Q}x + \Delta y_\ell \mathbf{Q}^t\partial_\ell\mathbf{Q}x + 2\mathbf{Q}^t\partial_\ell\mathbf{Q}\nabla y_\ell\big] + \sum_{\ell=1}^N \sum_{k=1}^N \big[\langle \partial_\ell\mathbf{Q}x, \partial_k^2\mathbf{Q}x\rangle + \Delta y_k \langle \partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}x\rangle + 2\langle \partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}\nabla y_k\rangle\big]\nabla y_\ell$.

Here ∂_ℓ, ∂_ℓ^2 denote the first and second derivatives with respect to y_ℓ, whereas the gradients and Laplacians of the variables y_ℓ, y_k are those with respect to $x_1, \ldots, x_n$.

Proof Recall that for a whirl we have $\mathscr{R}_u = |x|$, $\mathscr{S}_u = \mathbf{Q}(y)x|x|^{-1}$. With $\nabla\mathscr{R}_u = x|x|^{-1}$ identity (iii) follows at once from the corresponding identity in Lemma 1. For (i) referring to (3.8) we have $\nabla u = \mathscr{R}_u \nabla\mathscr{S}_u + \mathscr{S}_u \otimes \nabla\mathscr{R}_u$ and so

$$\nabla u = \mathbf{Q}\left(\mathbf{I}_n - \frac{x}{|x|} \otimes \frac{x}{|x|}\right) + \sum_{\ell=1}^N \partial_\ell\mathbf{Q}x \otimes \nabla y_\ell + \mathbf{Q}\frac{x}{|x|} \otimes \frac{x}{|x|} = \mathbf{Q} + \sum_{\ell=1}^N \partial_\ell\mathbf{Q}x \otimes \nabla y_\ell \tag{3.9}$$

as required. For $\mathbf{X}[u]$ we use $[\nabla u]^t[\nabla u] = \nabla\mathscr{R}_u \otimes \nabla\mathscr{R}_u + \mathscr{R}_u^2[\nabla\mathscr{S}_u]^t[\nabla\mathscr{S}_u]$ together with

$$\begin{aligned}
[\nabla\mathscr{S}_u]^t[\nabla\mathscr{S}_u] &= \frac{1}{|x|^2}\left[\mathbf{Q}^t - \frac{x \otimes \mathbf{Q}x}{|x|^2} + \sum_{\ell=1}^N \nabla y_\ell \otimes \partial_\ell\mathbf{Q}x\right]\left[\mathbf{Q} - \frac{\mathbf{Q}x \otimes x}{|x|^2} + \sum_{k=1}^N \partial_k\mathbf{Q}x \otimes \nabla y_k\right] \\
&= \frac{1}{|x|^2}\left[\mathbf{I}_n - x|x|^{-1} \otimes x|x|^{-1} + \sum_{\ell=1}^N [\mathbf{Q}^t\partial_\ell\mathbf{Q}x \otimes \nabla y_\ell + \nabla y_\ell \otimes \mathbf{Q}^t\partial_\ell\mathbf{Q}x]\right. \\
&\quad \left. + \sum_{\ell=1}^N \sum_{k=1}^N \langle \partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}x\rangle \nabla y_\ell \otimes \nabla y_k\right].
\end{aligned} \tag{3.10}$$

For $\mathbf{Y}[u]$ similarly we use $[\nabla u][\nabla u]^t = \mathscr{R}_u^2[\nabla\mathscr{S}_u][\nabla\mathscr{S}_u]^t + \mathscr{R}_u(\nabla\mathscr{S}_u\nabla\mathscr{R}_u \otimes \mathscr{S}_u + \mathscr{S}_u \otimes \nabla\mathscr{S}_u\nabla\mathscr{R}_u) + |\nabla\mathscr{R}_u|^2\mathscr{S}_u \otimes \mathscr{S}_u$ together with $\nabla\mathscr{S}_u\nabla\mathscr{R}_u = |x|^{-2}\sum_\ell \langle \nabla y_\ell, x\rangle \partial_\ell\mathbf{Q}x$ and hence

$$
\begin{aligned}
[\nabla\mathscr{S}_u][\nabla\mathscr{S}_u]^t &= \frac{1}{|x|^2}\left[\mathbf{Q} - \frac{\mathbf{Q}x\otimes x}{|x|^2} + \sum_{k=1}^N \partial_k\mathbf{Q}x\otimes\nabla y_k\right]\left[\mathbf{Q}^t - \frac{x\otimes\mathbf{Q}x}{|x|^2} + \sum_{\ell=1}^N \nabla y_\ell\otimes\partial_\ell\mathbf{Q}x\right] \\
&= \frac{1}{|x|^2}\left[\mathbf{I}_n - |x|^{-2}\mathbf{Q}x\otimes\mathbf{Q}x - |x|^{-2}\sum_{\ell=1}^N\langle\nabla y_\ell, x\rangle[\partial_\ell\mathbf{Q}x\otimes\mathbf{Q}x + \mathbf{Q}x\otimes\partial_\ell\mathbf{Q}x]\right. \\
&\quad \left. + \sum_{\ell=1}^N[\mathbf{Q}\nabla y_\ell\otimes\partial_\ell\mathbf{Q}x + \partial_\ell\mathbf{Q}x\otimes\mathbf{Q}\nabla y_\ell + \partial_\ell\mathbf{Q}x\otimes\partial_\ell\mathbf{Q}x]\right]
\end{aligned} \tag{3.11}
$$

which then gives $\mathbf{Y}[u]$. Note that here we have made use of the relation $\langle\nabla y_j, \nabla y_k\rangle = \delta_{jk}$. Next, regarding $|\nabla u|^2$ we have upon recalling (v) in Lemma 1,

$$
|\nabla u|^2 = \mathscr{R}_u^2|\nabla\mathscr{S}_u|^2 + |\nabla\mathscr{R}_u|^2 = |x|^2\left[\frac{n-1}{|x|^2} + \sum_{\ell=1}^N\frac{|\partial_\ell\mathbf{Q}x|^2}{|x|^2}\right] + 1 = n + \sum_{\ell=1}^N|\partial_\ell\mathbf{Q}x|^2.
$$

Some details are straightforward and hence omitted. Here and below we use the observation that $\langle\mathbf{Q}^t\partial_\ell\mathbf{Q}x, \nabla y_k\rangle = 0$ [see (4.7)]. For the determinant relation (ii) write $\det\nabla u = \det[\mathbf{Q} + \sum_{\ell=1}^N\partial_\ell\mathbf{Q}x\otimes\nabla y_\ell] = \det[\mathbf{I}_n + \sum_{\ell=1}^N\mathbf{Q}^t\partial_\ell\mathbf{Q}x\otimes\nabla y_\ell]$. Since for $p_i = \mathbf{Q}^t\partial_i\mathbf{Q}x$, $q_j = \nabla y_j$ we have $\langle p_i, q_j\rangle = 0$ for all $1\le i,j\le N$ (as above), it follows from Lemma 3.1 in [16] that $\det[\mathbf{I}_n + \sum_{j=1}^N p_j\otimes q_j] = 1$ which then gives (*ii*). Turning to the Laplacian we recall $\Delta\mathscr{R}_u = (n-1)/|x|$ and compute in a similar way

$$
\Delta\mathscr{S}_u = \frac{1-n}{|x|^3}\mathbf{Q}x + \frac{1}{|x|}\sum_{\ell=1}^N\left[2\partial_\ell\mathbf{Q}\nabla y_\ell - \frac{2}{|x|^2}\langle\nabla y_\ell, x\rangle\partial_\ell\mathbf{Q}x + \partial_\ell^2\mathbf{Q}x + \Delta y_\ell\partial_\ell\mathbf{Q}x\right]. \tag{3.12}
$$

As a result using $\Delta u = \mathscr{R}_u\Delta\mathscr{S}_u + 2\nabla\mathscr{S}_u\nabla\mathscr{R}_u + \Delta\mathscr{R}_u\mathscr{S}_u$ we thus obtain

$$
\begin{aligned}
\Delta u &= \sum_{\ell=1}^N\left[2\partial_\ell\mathbf{Q}\nabla y_\ell - \frac{2}{|x|^2}\langle\nabla y_\ell, x\rangle\partial_\ell\mathbf{Q}x + \partial_\ell^2\mathbf{Q}x + \Delta y_\ell\partial_\ell\mathbf{Q}x\right] - \frac{n-1}{|x|^2}\mathbf{Q}x \\
&\quad + \frac{2}{|x|^2}\sum_{\ell=1}^N\langle\nabla y_\ell, x\rangle\partial_\ell\mathbf{Q}x + \frac{n-1}{|x|^2}\mathbf{Q}x = \sum_{\ell=1}^N\left[\partial_\ell^2\mathbf{Q}x + \Delta y_\ell\partial_\ell\mathbf{Q}x + 2\partial_\ell\mathbf{Q}\nabla y_\ell\right].
\end{aligned}
$$

A calculation using ingredients already gathered in the proof also verifies (viii).

Proposition 3 *Let u be a whirl map with $\mathbf{Q}\in\mathscr{C}(\overline{\mathbb{A}}_n, \mathbf{SO}(n))\cap\mathscr{C}^2(\mathbb{A}_n, \mathbf{SO}(n))$. Then the action of the partial differential operator $\mathscr{L}$ on u can be described in terms of $\mathbf{Q}$ as*

$$\begin{aligned}
\mathscr{L}[u] - \nabla F_\xi = 2F_{\xi\xi}\Bigg\{ & \sum_{\ell=1}^N \sum_{k=1}^N [\langle \partial_{\ell k}^2 \mathbf{Q}x, \partial_\ell \mathbf{Q}x\rangle + \langle \nabla y_k, \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q}x\rangle]\mathbf{Q}^t \partial_k \mathbf{Q}x \\
& + \sum_{\ell=1}^N \sum_{k=1}^N \sum_{j=1}^N [\langle \partial_{\ell k}^2 \mathbf{Q}x, \partial_\ell \mathbf{Q}x\rangle + \langle \nabla y_k, \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q}x\rangle]\langle \partial_j \mathbf{Q}x, \partial_k \mathbf{Q}x\rangle \nabla y_j \Bigg\} \\
& + \Big[2F_{s\xi} + |x|^{-1} F_{r\xi}\Big]\Bigg\{ \sum_{\ell=1}^N \langle \nabla y_\ell, x\rangle \mathbf{Q}^t \partial_\ell \mathbf{Q}x + \sum_{\ell=1}^N \sum_{k=1}^N \langle \partial_\ell \mathbf{Q}x, \partial_k \mathbf{Q}x\rangle \langle \nabla y_k, x\rangle \nabla y_\ell \Bigg\} \\
& + F_\xi \Bigg\{ \sum_{\ell=1}^N \big[\mathbf{Q}^t \partial_\ell^2 \mathbf{Q}x + \Delta y_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q}x + 2\mathbf{Q}^t \partial_\ell \mathbf{Q} \nabla y_\ell\big] \\
& + \sum_{\ell=1}^N \sum_{k=1}^N \big[\langle \partial_\ell \mathbf{Q}x, \partial_k^2 \mathbf{Q}x\rangle + \Delta y_k \langle \partial_\ell \mathbf{Q}x, \partial_k \mathbf{Q}x\rangle + 2\langle \partial_\ell \mathbf{Q}x, \partial_k \mathbf{Q}\nabla y_k\rangle\big] \nabla y_\ell \Bigg\} - F_s x.
\end{aligned} \tag{3.13}$$

The arguments of $F = F(r, s, \xi)$ *in* (3.13) *and all subsequent derivatives are* $(r, s, \xi) = (|x|, |u|^2, |\nabla u|^2) = (r, r^2, n + \sum_{\ell=1}^N |\partial_\ell \mathbf{Q}x|^2)$.

Proof We use Proposition 1 and similar to the argument in Proposition 2 proceed by computing the various coefficients associated with $\mathscr{L}[u]$ as described by (2.4)–(2.7). Indeed for $\mathsf{A}[u]$ we have

$$\begin{aligned}
\mathsf{A}[u] &= \big[\mathscr{R}_u^2 [\nabla \mathscr{S}_u]^t [\nabla \mathscr{S}_u] + \nabla \mathscr{R}_u \otimes \nabla \mathscr{R}_u\big] \nabla \Big[\mathscr{R}_u^2 |\nabla \mathscr{S}_u|^2 + |\nabla \mathscr{R}_u|^2\Big] \\
&= \Bigg[\mathbf{I}_n + \sum_{\ell=1}^N [\mathbf{Q}^t \partial_\ell \mathbf{Q}x \otimes \nabla y_\ell + \nabla y_\ell \otimes \mathbf{Q}^t \partial_\ell \mathbf{Q}x] \\
&\quad + \sum_{\ell=1}^N \sum_{k=1}^N \langle \partial_\ell \mathbf{Q}x, \partial_k \mathbf{Q}x\rangle \nabla y_\ell \otimes \nabla y_k \Bigg] \nabla \Bigg[n + \sum_{\ell=1}^N |\partial_\ell \mathbf{Q}x|^2\Bigg]
\end{aligned} \tag{3.14}$$

where for the last term on the right we have

$$\begin{aligned}
\nabla \Bigg[n + \sum_{\ell=1}^N |\partial_\ell \mathbf{Q}x|^2\Bigg] &= \nabla \sum_{\ell=1}^N |\partial_\ell \mathbf{Q}x|^2 = \nabla \Bigg\langle \sum_{\ell=1}^N \partial_\ell \mathbf{Q}x, \sum_{\ell=1}^N \partial_\ell \mathbf{Q}x \Bigg\rangle \\
&= 2\sum_{\ell=1}^N \Bigg[\sum_{k=1}^N [\nabla y_k \otimes \partial_{\ell k}^2 \mathbf{Q}x]\partial_\ell \mathbf{Q}x + \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q}x\Bigg] \\
&= 2\sum_{\ell=1}^N \Bigg[\sum_{k=1}^N \langle \partial_{\ell k}^2 \mathbf{Q}x, \partial_\ell \mathbf{Q}x\rangle \nabla y_k + \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q}x\Bigg].
\end{aligned} \tag{3.15}$$

Regarding the vector field $\mathsf{B}[u]$ we have

$$\begin{aligned}
\mathsf{B}[u] &= 2\mathscr{R}_u \Big[\mathscr{R}_u^2 [\nabla \mathscr{S}_u]^t [\nabla \mathscr{S}_u] \nabla \mathscr{R}_u + |\nabla \mathscr{R}_u|^2 \nabla \mathscr{R}_u\Big] \\
&= 2x + 2\sum_{\ell=1}^N \Bigg[\langle \nabla y_\ell, x\rangle \mathbf{Q}^t \partial_\ell \mathbf{Q}x + \sum_{k=1}^N \langle \partial_\ell \mathbf{Q}x, \partial_k \mathbf{Q}x\rangle \langle \nabla y_k, x\rangle \nabla y_\ell\Bigg].
\end{aligned} \tag{3.16}$$

For $\mathsf{C}[u]$ the calculation is again similar and we have

$$\begin{aligned}\mathsf{C}[u] =& \left[\mathscr{R}_u^2[\nabla\mathscr{S}_u]^t[\nabla\mathscr{S}_u] + \nabla\mathscr{R}_u \otimes \nabla\mathscr{R}_u\right]x|x|^{-1}\\ &= x|x|^{-1} + \sum_{\ell=1}^{N}\left[\langle\nabla y_\ell, x\rangle\mathbf{Q}^t\partial_\ell\mathbf{Q}x + \sum_{k=1}^{N}\langle\partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}x\rangle\langle\nabla y_k, x\rangle\nabla y_\ell\right]|x|^{-1}.\end{aligned} \tag{3.17}$$

Finally for $\mathsf{D}[u] = [\nabla u]^t\Delta u$ and $\mathsf{E}[u] = -[\nabla u]^t u$ we refer to (viii) and (iii) in Lemma 3. Putting these together and noting that $\nabla F_\xi = F_{\xi\xi}\nabla|\nabla u|^2 + (2F_{s\xi} + |x|^{-1}F_{r\xi})x$ gives the desired conclusion.

4 Derivation of the relation between the constrained and unconstrained operators

We now consider restricting the energy integral $\mathbb{I}$ to the subclass of admissible whirls hence obtaining a restricted energy integral (called $\mathbb{H}$ below) for the angle of rotation vector function as in (3.6)–(3.7). Our efforts then shift to carefully analysing the resulting Euler–Lagrange equation, its solvability, and most notably uncovering the relation it bears to the original Euler–Lagrange system (1.2). Towards this end we begin by writing the energy of a whirl map u associated with the matrix-field $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ as [cf. (*vi*) in Lemma 3 and (4.7) below]

$$\begin{aligned}\mathbb{I}[u;\mathbb{X}_n] &= \int_a^b\int_{\mathbb{S}^{n-1}} F\left(r, r^2, n + \sum_{\ell=1}^{N}|\partial_\ell\mathbf{Q}x|^2\right)r^{n-1}\,dr d\mathcal{H}^{n-1}(\theta)\\ &= (2\pi)^d\int_{\mathbb{A}_n} F\left(||y||, ||y||^2, n + \sum_{j=1}^{d} y_j^2|\nabla f_j|^2\right)\mathscr{J}(y)\,dy =: (2\pi)^d\mathbb{H}[\mathsf{f};\mathbb{A}_n].\end{aligned} \tag{4.1}$$

Here $\mathsf{J} = (2\pi)^d\mathscr{J}(y)$ is the Jacobian for the change of variables from $x = (x_1, \ldots, x_n)$ to $y = (y_1, \ldots, y_N)$ with $\mathscr{J}(y) = y_1\cdots y_d$. In particular note that when $n = 2d$ is even $\mathscr{J}(y) = y_1\ldots y_N$ whereas when $n = 2d+1$ is odd $\mathscr{J}(y) = y_1\ldots y_{N-1}$ and so the last variable y_N does not appear in this Jacobian product. One important implication here is that $\mathscr{J}$ is always *strictly* positive in $\mathbb{A}_n$. Moving forward we now aim to extremise $\mathbb{H}[\mathsf{f};\mathbb{A}_n]$ over the space of admissible vector functions $\mathscr{B}^p_{\mathsf{m}}[\mathbb{A}_n]$, defined for $\mathsf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ and $p \geq 1$, by $\mathscr{B}^p_{\mathsf{m}}[\mathbb{A}_n] = \{\mathsf{f} \in W^{1,p}(\mathbb{A}_n, \mathbb{R}^d) : \mathsf{f} \equiv 0 \text{ on } (\partial\mathbb{A}_n)_a, \mathsf{f} \equiv 2\mathsf{m}\pi \text{ on } (\partial\mathbb{A}_n)_b\}$. It is not difficult to see that the resulting Euler–Lagrange system here takes the form (for $1 \leq i \leq d$, $\mathsf{m} \in \mathbb{Z}^d$)

$$\mathbf{RS}[\mathsf{f},\mathbb{A}_n] = \begin{cases}\operatorname{div}[\mathsf{A}_i(y,\nabla\mathsf{f})\nabla f_i] = 0 & \text{in } \mathbb{A}_n,\\ \mathsf{f} \equiv 0 & \text{on } (\partial\mathbb{A}_n)_a,\\ \mathsf{f} \equiv 2\mathsf{m}\pi & \text{on } (\partial\mathbb{A}_n)_b,\\ \mathsf{A}_i(y,\nabla\mathsf{f})\partial_\nu f_i = 0 & \text{on } \Gamma_n,\end{cases} \tag{4.2}$$

where for $1 \le i \le d$, we have set,

$$\mathsf{A}_i(y, \nabla \mathsf{f}) = F_\xi \left(||y||, ||y||^2, n + \sum_{j=1}^{d} y_j^2 |\nabla f_j|^2 \right) y_i^2 \mathscr{I}(y), \qquad ||y||^2 = \sum_{j=1}^{N} y_j^2. \quad (4.3)$$

Theorem 1 *For each* $\mathsf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$ *the solution* $\mathsf{f} \in \mathscr{C}^2(\overline{\mathbb{A}}_n, \mathbb{R}^d)$ *of the system* $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ *is unique.*

Proof Let us fix a solution f as described and set $\mathsf{g} = \mathsf{f} + \varphi$, $\mathsf{g} \in \mathscr{B}^p_{\mathsf{m}}[\mathbb{A}_n]$. Then evidently $F(z, z^2, \zeta_2) - F(z, z^2, \zeta_1) \ge F_\xi(z, z^2, \zeta_1)(\zeta_2 - \zeta_1)$ for $z \in [a, b]$, $\zeta_1, \zeta_2 \in \mathbb{R}$. Thus taking $\zeta_1 = n + \sum y_j^2 |\nabla f_j|^2$, $\zeta_2 = n + \sum y_j^2 |\nabla g_j|^2$ and noting $\varphi \equiv 0$ on $(\partial \mathbb{A}_n)_a \cup (\partial \mathbb{A}_n)_b$ we have (upon writing $F_\xi = F_\xi(z, z^2, n + \sum_{j=1}^d y_j^2 |\nabla f_j|^2)$, $z^2 = ||y||^2 = \sum_{j=1}^N y_j^2$ for brevity)

$$\begin{aligned}
\mathbb{H}[\mathsf{g}; \mathbb{A}_n] - \mathbb{H}[\mathsf{f}, \mathbb{A}_n] &\ge \int_{\mathbb{A}_n} F_\xi \sum_{\alpha=1}^{d} y_\alpha^2 (|\nabla g_\alpha|^2 - |\nabla f_\alpha|^2) \mathscr{I}(y) \, \mathrm{d}y \\
&\ge 2 \sum_{\alpha=1}^{d} \int_{\mathbb{A}_n} -\mathrm{div} \left[F_\xi y_\alpha^2 \nabla f_\alpha \mathscr{I}(y) \right] \varphi_\alpha \, \mathrm{d}y + 2 \sum_{\alpha=1}^{d} \int_{\Gamma_n} \left[F_\xi y_\alpha^2 \mathscr{I}(y) \partial_\nu f_\alpha \right] \varphi_\alpha \, \mathrm{d}y \\
&\quad + \int_{\mathbb{A}_n} F_\xi \sum_{\alpha=1}^{d} y_\alpha^2 |\nabla \varphi_\alpha|^2 \mathscr{I}(y) \, \mathrm{d}y \ge \int_{\mathbb{A}_n} F_\xi \sum_{\alpha=1}^{d} y_\alpha^2 |\nabla \varphi_\alpha|^2 \mathscr{I}(y) \, \mathrm{d}y.
\end{aligned} \quad (4.4)$$

Here, we are using $|\nabla g_\alpha|^2 - |\nabla f_\alpha|^2 = |\nabla (g_\alpha - f_\alpha)|^2 + 2 \nabla f_\alpha \cdot \nabla (g_\alpha - f_\alpha)$ and the fact that f is a solution to (4.2) to deduce that the first and second integrals on the second line in (4.4) vanish. In particular the quantity on the left is non-negative giving the minimality of f in $\mathscr{B}^p_{\mathsf{m}}[\mathbb{A}_n]$. Now if f, g are both solutions as described in $\mathscr{B}^p_{\mathsf{m}}[\mathbb{A}_n]$ then arguing as above $\mathbb{H}[\mathsf{g}; \mathbb{A}_n] = \mathbb{H}[\mathsf{f}, \mathbb{A}_n]$. Thus the integral on the right in (4.4) vanishes and so in view of the strict inequalities $F_\xi, \mathscr{I} > 0$ in $\mathbb{A}_n$ it follows at once that $\mathsf{f} \equiv \mathsf{g}$.

To proceed forward we next introduce the collection of $2N$ orthogonal n-vectors: $w^i = (0, \dots, 0, x_{2i-1}, x_{2i}, 0, \dots, 0)$, $[w^i]^\perp = (0, \dots, 0, -x_{2i}, x_{2i-1}, 0, \dots, 0)$ for $1 \le i \le d$; when $n = 2d$ is even this completes the picture, when $n = 2d + 1$ is odd we set $w^N = (0, \dots, 0, x_n)$, $[w^N]^\perp = (0, \dots, 0)$. Hence $x = \sum_{i=1}^N w^i$, $\langle w^i, w^j \rangle = 0$, $\langle [w^i]^\perp, [w^j]^\perp \rangle = 0$ for $1 \le i \ne j \le N$ and $\langle w^i, [w^j]^\perp \rangle = 0$ for all $1 \le i, j \le N$, whilst in relation to the variables $y_1, \dots, y_N$ introduced earlier in Sect. 2, we have $y_\ell = |w^\ell| = |[w^\ell]^\perp|$ when $1 \le \ell \le d$ noting that when $n = 2d + 1$ we have $w^N = (0, \dots, 0, y_N)$ and $|y_N| = |w^N| = |x_n|$. Evidently $\nabla y_\ell = w^\ell / y_\ell$.

Proposition 4 *Let u be a whirl map associated with the matrix field* $\mathbf{Q} = \mathbf{Q}[\mathsf{f}](y)$ *of class* $\mathscr{C}^2(\overline{\mathbb{A}}_n, \mathbf{SO}(n))$. *Then with* $\mathsf{A}_i = \mathsf{A}_i(y, \nabla \mathsf{f})$ *as in* (4.3) *and* w^i, $[w^i]^\perp$ *as above we have*

$$
\begin{aligned}
\sum_{i=1}^{d} \mathrm{div}[\mathsf{A}_i \nabla f_i] \frac{[w^i]^\perp}{\mathscr{J} y_i^2} = & \sum_{\ell=1}^{N} \Big\{ 2F_{\xi\xi} \sum_{k=1}^{N} \big[\langle \nabla y_\ell, \partial_k \mathbf{Q}^t \partial_k \mathbf{Q} x \rangle + \langle \partial^2_{\ell k} \mathbf{Q} x, \partial_k \mathbf{Q} x \rangle \big] \mathbf{Q}^t \partial_\ell \mathbf{Q} x \\
& + [2F_{s\xi} + |x|^{-1} F_{r\xi}] \langle \nabla y_\ell, x \rangle \mathbf{Q}^t \partial_\ell \mathbf{Q} x \\
& + F_\xi \big[\Delta y_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q} x + \mathbf{Q}^t \partial^2_\ell \mathbf{Q} x + 2 \mathbf{Q}^t \partial_\ell \mathbf{Q} \nabla y_\ell \big] + F_\xi \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q} x \Big\}.
\end{aligned}
\tag{4.5}
$$

Proof Starting from the expression on the left-hand side and expanding the divergence term on the left, a straightforward differentiation gives

$$
\begin{aligned}
\frac{1}{\mathscr{J} y_i^2} \mathrm{div}[\mathsf{A}_i(y, \nabla \mathsf{f}) \nabla f_i] &= \frac{1}{\mathscr{J} y_i^2} \sum_{k=1}^{N} \partial_k \left[F_\xi \left(z, z^2, n + \sum_{j=1}^{d} y_j^2 |\nabla f_j|^2 \right) y_i^2 y_1 \cdots y_d \partial_k f_i \right] \\
&= \sum_{k=1}^{d} \sum_{\ell=1}^{N} 2F_{\xi\xi} y_k (\partial_\ell f_k)^2 \partial_k f_i + \sum_{k=1}^{N} \left[\sum_{j=1}^{d} \sum_{\ell=1}^{N} 2F_{\xi\xi} y_j^2 \partial^2_{\ell k} f_j \partial_\ell f_j \partial_k f_i \right. \\
&\quad \left. + y_k [2F_{s\xi} + |x|^{-1} F_{r\xi}] \partial_k f_i + F_\xi \partial_k^2 f_i \right] + \sum_{k=1}^{d} y_k^{-1} F_\xi \partial_k f_i + 2 y_i^{-1} F_\xi \partial_i f_i.
\end{aligned}
\tag{4.6}
$$

Here, we have made use of the relation

$$
\frac{1}{\mathscr{J} y_i^2} \sum_{k=1}^{N} \partial_k (y_i^2 y_1 \cdots y_d \partial_k f_i) = 2 y_i^{-1} \partial_i f_i + \sum_{k=1}^{d} y_k^{-1} \partial_k f_i + \sum_{k=1}^{N} \partial_k^2 f_i, \quad 1 \le i \le d.
$$

Regarding the expressions on the right-hand side of (4.5) and by evaluating the individual terms it can be seen that (below $1 \le \ell \le N$ and $1 \le k \le N$)

$$
\begin{cases}
\mathbf{Q}^t \partial_\ell \mathbf{Q} \nabla y_k = y_k^{-1} \partial_\ell f_k [w^k]^\perp, & \mathbf{Q}^t \partial_\ell \mathbf{Q} x = \sum_{i=1}^{d} \partial_\ell f_i [w^i]^\perp, \\
\partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q} x = \sum_{i=1}^{d} (\partial_\ell f_i)^2 w^i, & \mathbf{Q}^t \partial^2_{\ell k} \mathbf{Q} x = \sum_{i=1}^{d} \left[\partial^2_{\ell k} f_i [w^i]^\perp - \partial_\ell f_i \partial_k f_i w^i \right].
\end{cases}
\tag{4.7}
$$

The third identity above along with the inner product relation $\langle \nabla y_\ell, w^i \rangle = y_\ell \delta_{i\ell}$ leads to $\langle \nabla y_\ell, \partial_k \mathbf{Q}^t \partial_k \mathbf{Q} x \rangle = y_\ell (\partial_k f_\ell)^2$ when $1 \le \ell \le d$ (and zero when n is odd and $\ell = N$), whilst using the third and fourth identities we can obtain

$$
\sum_{\ell=1}^{N} \left[\mathbf{Q}^t \partial_\ell^2 \mathbf{Q} x + \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q} x \right] = \sum_{i=1}^{d} \sum_{\ell=1}^{N} \partial_\ell^2 f_i [w^i]^\perp,
\tag{4.8}
$$

$$
\sum_{\ell=1}^{N} \left[\Delta y_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q} x + 2 \mathbf{Q}^t \partial_\ell \mathbf{Q} \nabla y_\ell \right] = \sum_{i=1}^{d} \left[\sum_{\ell=1}^{N} \Delta y_\ell \partial_\ell f_i + 2 y_i^{-1} \partial_i f_i \right] [w^i]^\perp.
\tag{4.9}
$$

We note that the above expressions (specifically the sums on the left-hand side in the two identities above) form precisely the coefficients of F_ξ in (4.5). In much the same way using the second and fourth identities in (4.7) lead to

$$\langle \partial^2_{\ell k}\mathbf{Q}x, \partial_k \mathbf{Q}x\rangle = \langle \mathbf{Q}^t\partial^2_{\ell k}\mathbf{Q}x, \mathbf{Q}^t\partial_k\mathbf{Q}x\rangle = \left\langle \sum_{i=1}^d \left[\partial^2_{\ell k}f_i[w^i]^\perp - \partial_\ell f_i\partial_k f_i w^i\right], \sum_{j=1}^d \partial_k f_j [w^j]^\perp \right\rangle$$
$$= \sum_{i=1}^d\sum_{j=1}^d \left[\partial^2_{\ell k}f_i\partial_k f_j\langle [w^i]^\perp, [w^j]^\perp\rangle - \partial_\ell f_i\partial_k f_i\partial_k f_j\langle w^i, [w^j]^\perp\rangle\right] = \sum_{i=1}^d y_i^2\partial^2_{\ell k}f_i\partial_k f_i, \tag{4.10}$$

where use has been made of the orthogonality relations $\langle w^i, [w^j]^\perp\rangle = 0$ $(1\le i,j\le N)$ along with $\langle [w^i]^\perp, [w^j]^\perp\rangle = 0$ $(i\ne j)$ and $\langle [w^j]^\perp, [w^j]^\perp\rangle = y_j^2$. As a result we can write

$$\sum_{\ell=1}^N\sum_{k=1}^N \left[\langle \nabla y_\ell, \partial_k\mathbf{Q}^t\partial_k\mathbf{Q}x\rangle + \langle \partial^2_{\ell k}\mathbf{Q}x, \partial_k\mathbf{Q}x\rangle\right]\mathbf{Q}^t\partial_\ell\mathbf{Q}x$$
$$= \sum_{i=1}^d\sum_{\ell=1}^N\sum_{k=1}^N \left[y_\ell(\partial_k f_\ell)^2 + \sum_{j=1}^d y_j^2\partial^2_{\ell k}f_j\partial_k f_j\right]\partial_\ell f_i[w^i]^\perp, \tag{4.11}$$

which forms the coefficient of $2F_{\xi\xi}$ in (4.5). Substituting back the relevant terms and taking into account the cancellations gives the required relation.

This brings us to the following result bridging the differential operator action $\mathscr{L}[u]$ for a whirl u associated with the matrix-field $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ and the system $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$.

Theorem 2 *Let u be a whirl map associated with the matrix field $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ where $\mathsf{f} = (f_1,\dots,f_d)$ is of class $\mathscr{C}^2(\overline{\mathbb{A}}_n, \mathbb{R}^d)$ [see (3.6)–(3.7)]. Then the constrained PDE system $\Sigma[(u,\mathscr{P},\Omega)]$ and the unconstrained system $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ are directly related to one-another via the identity*

$$\mathscr{L}[u] = \nabla F_\xi - F_s x - \sum_{\ell=1}^N F_\xi\partial_\ell\mathbf{Q}^t\partial_\ell\mathbf{Q}x$$
$$+ \frac{1}{\mathscr{J}(y)}\left[\mathbf{I}_n + \sum_{\ell=1}^N \nabla y_\ell\otimes\mathbf{Q}^t\partial_\ell\mathbf{Q}x\right]\left\{\sum_{i=1}^d \frac{1}{y_i^2}\mathrm{div}[\mathsf{A}_i(y,\nabla\mathsf{f})\nabla f_i][w^i]^\perp\right\}. \tag{4.12}$$

The arguments of $F_s = F_s(r,s,\xi)$ and $F_\xi = F_\xi(r,s,\xi)$ are $(r,s,\xi) = (|x|, |u|^2, |\nabla u|^2)$ and the coefficients $\mathsf{A}_i = \mathsf{A}_i(y,\nabla\mathsf{f})$ in (4.12) are exactly as given by (4.3).

Proof Starting with the description of $\mathscr{L}[u] - \nabla F_\xi$ as in (3.13) a close inspection reveals

$$\begin{aligned}
&\sum_{\ell=1}^{N}\sum_{k=1}^{N}\left[\langle \nabla y_\ell, \partial_k \mathbf{Q}^t \partial_k \mathbf{Q}x\rangle + \langle \partial^2_{\ell k}\mathbf{Q}x, \partial_k \mathbf{Q}x\rangle\right]\mathbf{Q}^t\partial_\ell \mathbf{Q}x \\
&\quad + \sum_{\ell=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\left[\langle \nabla y_k, \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q}x\rangle + \langle \partial^2_{\ell k}\mathbf{Q}x, \partial_\ell \mathbf{Q}x\rangle\right]\langle \partial_j \mathbf{Q}x, \partial_k \mathbf{Q}x\rangle \nabla y_j \\
&= \sum_{\ell=1}^{N}\sum_{k=1}^{N}\left[\langle \nabla y_\ell, \partial_k \mathbf{Q}^t \partial_k \mathbf{Q}x\rangle + \langle \partial^2_{\ell k}\mathbf{Q}x, \partial_k \mathbf{Q}x\rangle\right]\mathbf{Q}^t\partial_\ell \mathbf{Q}x \\
&\quad + \left\langle \sum_{k=1}^{N}\mathbf{Q}^t\partial_k\mathbf{Q}x, \sum_{\ell=1}^{N}\sum_{k=1}^{N}\left[\langle \nabla y_\ell, \partial_k \mathbf{Q}^t \partial_k \mathbf{Q}x\rangle + \langle \partial^2_{\ell k}\mathbf{Q}x, \partial_k \mathbf{Q}x\rangle\right]\mathbf{Q}^t\partial_\ell \mathbf{Q}x \right\rangle \nabla y_\ell \\
&= \left[\mathbf{I}_n + \sum_{\ell=1}^{N}\nabla y_\ell \otimes \mathbf{Q}^t\partial_\ell\mathbf{Q}x\right]\sum_{\ell=1}^{N}\sum_{k=1}^{N}\left[\langle \nabla y_\ell, \partial_k \mathbf{Q}^t \partial_k \mathbf{Q}x\rangle + \langle \partial^2_{\ell k}\mathbf{Q}x, \partial_k \mathbf{Q}x\rangle\right]\mathbf{Q}^t\partial_\ell \mathbf{Q}x.
\end{aligned} \tag{4.13}$$

Likewise moving to the next set of terms in (3.13) it is seen that

$$\begin{aligned}
&\sum_{\ell=1}^{N}\langle \nabla y_\ell, x\rangle \mathbf{Q}^t\partial_\ell\mathbf{Q}x + \sum_{\ell=1}^{N}\sum_{k=1}^{N}\langle \partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}x\rangle\langle \nabla y_k, x\rangle \nabla y_\ell \\
&\quad = \left[\mathbf{I}_n + \sum_{\ell=1}^{N}\nabla y_\ell \otimes \mathbf{Q}^t\partial_\ell\mathbf{Q}x\right]\sum_{\ell=1}^{N}\langle \nabla y_\ell, x\rangle \mathbf{Q}^t\partial_\ell\mathbf{Q}x,
\end{aligned} \tag{4.14}$$

and in a similar way

$$\begin{aligned}
&\sum_{\ell=1}^{N}\left[\Delta y_\ell \mathbf{Q}^t\partial_\ell\mathbf{Q}x + \mathbf{Q}^t\partial^2_\ell\mathbf{Q}x + 2\mathbf{Q}^t\partial_\ell\mathbf{Q}\nabla y_\ell\right] \\
&\qquad + \sum_{\ell=1}^{N}\sum_{k=1}^{N}\left[\langle \partial_\ell\mathbf{Q}x, \partial^2_k\mathbf{Q}x\rangle + \Delta y_k\langle \partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}x\rangle + 2\langle \partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}\nabla y_k\rangle\right]\nabla y_\ell \\
&\quad = \sum_{\ell=1}^{N}\left[\Delta y_\ell \mathbf{Q}^t\partial_\ell\mathbf{Q}x + \mathbf{Q}^t\partial^2_\ell\mathbf{Q}x + 2\mathbf{Q}^t\partial_\ell\mathbf{Q}\nabla y_\ell\right] \\
&\qquad + \left\langle \sum_{k=1}^{N}\mathbf{Q}^t\partial_k\mathbf{Q}x, \sum_{\ell=1}^{N}\left[\Delta y_\ell \mathbf{Q}^t\partial_\ell\mathbf{Q}x + \mathbf{Q}^t\partial^2_\ell\mathbf{Q}x + 2\mathbf{Q}^t\partial_\ell\mathbf{Q}\nabla y_\ell\right]\right\rangle \nabla y_\ell \\
&\quad = \left[\mathbf{I}_n + \sum_{\ell=1}^{N}\nabla y_\ell \otimes \mathbf{Q}^t\partial_\ell\mathbf{Q}x\right]\sum_{\ell=1}^{N}\left[\Delta y_\ell \mathbf{Q}^t\partial_\ell\mathbf{Q}x + \mathbf{Q}^t\partial^2_\ell\mathbf{Q}x + 2\mathbf{Q}^t\partial_\ell\mathbf{Q}\nabla y_\ell\right].
\end{aligned} \tag{4.15}$$

Recalling (4.5) and noting that by skew-symmetry $\langle \mathbf{Q}^t\partial_\ell\mathbf{Q}x, \partial_k\mathbf{Q}^t\partial_k\mathbf{Q}x\rangle = 0$, (4.12) follows by putting the above fragments together and rearranging terms.

Corollary 1 *Under the assumptions of the previous theorem suppose that the vector function* $\mathsf{f} = (f_1, \ldots, f_d)$ *solves the restricted system* $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$. *Then denoting by* $u = \mathbf{Q}[\mathsf{f}](x)$ *the whirl map associated with the matrix-field* $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ *we have*

$$\mathscr{L}[u] - \nabla F_\xi = -\sum_{\ell=1}^{N} F_\xi \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q} x - F_s x. \tag{4.16}$$

Proof As $\operatorname{div}[\mathsf{A}_i(y, \nabla \mathsf{f}) \nabla f_i] = 0$ for each $1 \le i \le d$ the assertion follows from (4.12).

5 The operator $\mathscr{L}[u]$ and a gradient-curl analysis for 2-plane n-vector fields

Returning to the Euler–Lagrange system $\Sigma[(u, \mathscr{P}), \Omega]$ we now aim to discuss the solvability of this system by taking advantage of the results of the previous section. Recall that if a whirl u associated with the $\mathbf{SO}(n)$-valued matrix field $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ is a solution to $\Sigma[(u, \mathscr{P}), \Omega]$ then the vector function $\mathsf{f} = (f_1, \ldots, f_d)$ is in turn a solution to the reduced system $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$. Hence in light of Corollary 1 we can write

$$\begin{aligned} \mathscr{L}[u] &= [\nabla u]^t \Big\{ \operatorname{div}\Big[F_\xi(|x|, |u|^2, |\nabla u|^2) \nabla u \Big] - F_s(|x|, |u|^2, |\nabla u|^2) u \Big\} \\ &= \nabla F_\xi - \sum_{\ell=1}^{N} F_\xi \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q} x - F_s x = \nabla F_\xi - \sum_{i=1}^{d} F_\xi |\nabla f_i|^2 w^i - \sum_{j=1}^{N} F_s w^j. \end{aligned} \tag{5.1}$$

Proposition 5 *Consider the vector field* $V = V(x)$ *of class* $\mathscr{C}^1(\mathbb{X}_n, \mathbb{R}^n)$ *defined by*

$$V(x) = \sum_{p=1}^{N} \alpha_p(y) w^p + \sum_{q=1}^{N} \beta_q(y) [w^q]^\perp. \tag{5.2}$$

Here $y = y(x) = (y_1, \ldots, y_N) \in \mathbb{A}_n$, $x = (x_1, \ldots, x_n) \in \mathbb{X}_n$, $w^p = w^p(x)$, $[w^q]^\perp = [w^q(x)]^\perp$ *and* $\alpha_p = \alpha_p(y)$, $\beta_q = \beta_q(y) \in \mathscr{C}^1(\mathbb{A}_n)$ *for all* $1 \le p, q \le N$. *Then*

$$\begin{aligned} \operatorname{curl} V = \nabla V - [\nabla V]^t &= \sum_{p=1}^{N} \sum_{k=1}^{N} \frac{\partial_k \alpha_p}{y_k} \left[w^p \otimes w^k - w^k \otimes w^p \right] \\ &\quad + \sum_{q=1}^{N} \sum_{k=1}^{N} \frac{\partial_k \beta_q}{y_k} \left[[w^q]^\perp \otimes w^k - w^k \otimes [w^q]^\perp \right] + \sum_{q=1}^{d} 2\beta_q \Lambda^q. \end{aligned} \tag{5.3}$$

Here $\Lambda^q = \operatorname{diag}(0, \ldots, 0, \mathbf{J}, 0 \ldots, 0)$, *that is, the* $n \times n$ *skew-symmetric block diagonal matrix with* $\mathbf{J}$ *its* q *th block.*[5]

[5] Thus $\Lambda^q_{ij} = -1$ when $i = 2q-1, j = 2q$, $\Lambda^q_{ij} = +1$ when $i = 2q, j = 2q-1$ and $\Lambda^q_{ij} = 0$ otherwise.

Proof We verify this by direct evaluation of the curl. Indeed for $1 \le i < j \le n$ we have (with w_i^p denoting the i component of w^p and $[w_i^q]^\perp$ denoting the i component or $[w^q]^\perp$)

$$\begin{aligned}
[\mathrm{curl}\, V]_{ij} &= \frac{\partial V_i}{\partial x_j} - \frac{\partial V_j}{\partial x_i} \\
&= \sum_{p=1}^N \sum_{k=1}^N \partial_k \alpha_p \frac{w_j^k}{y_k} w_i^p + \sum_{p=1}^N \alpha_p \partial_j w_i^p + \sum_{q=1}^N \sum_{k=1}^N \partial_k \beta_q \frac{w_j^k}{y_k} [w_i^q]^\perp + \sum_{q=1}^N \beta_q \partial_j [w_i^q]^\perp \\
&\quad - \sum_{p=1}^N \sum_{k=1}^N \partial_k \alpha_p \frac{w_i^k}{y_k} w_j^p - \sum_{p=1}^N \alpha_p \partial_i w_j^p - \sum_{q=1}^N \sum_{k=1}^N \partial_k \beta_q \frac{w_i^k}{y_k} [w_j^q]^\perp - \sum_{q=1}^N \beta_q \partial_i [w_j^q]^\perp \\
&= \sum_{p=1}^N \sum_{k=1}^N \frac{\partial_k \alpha_p}{y_k} \left(w_j^k w_i^p - w_i^k w_j^p \right) + \sum_{p=1}^N \alpha_p \left(\partial_j w_i^p - \partial_i w_j^p \right) \\
&\quad + \sum_{q=1}^N \sum_{k=1}^N \frac{\partial_k \beta_q}{y_k} \left(w_i^k [w_i^q]^\perp - w_i^k [w_j^q]^\perp \right) + \sum_{q=1}^N \beta_q \left(\partial_j [w_i^q]^\perp - \partial_i [w_j^q]^\perp \right).
\end{aligned} \tag{5.4}$$

By inspection $[\mathrm{curl}\, w^p]_{ij} = \partial_j w_i^p - \partial_i w_j^p \equiv 0$, $[\mathrm{curl}\, [w^q]^\perp]_{ij} = \partial_j [w_i^q]^\perp - \partial_i [w_j^q]^\perp \equiv 2\Lambda_{ij}^q$ and so the conclusion follows at once by substitution.

Before discussing applications of the proposition to Theorem 2 let us pause briefly to take a closer look at some special cases of the statement and its implications.

- If α_p $(1 \le p \le N)$, β_q $(1 \le q \le d)$ are constant (in y) then $\mathrm{curl}\, V = 2 \sum_{q=1}^d \beta_q \Lambda^q$ and so in particular $\mathrm{curl}\, V \equiv 0 \iff \beta_q \equiv 0$ for all q.
- If $\alpha_p(y) = \alpha_p(y_p)$, $\beta_q(y) = \beta_q(y_q)$ for all $1 \le p, q \le N$ then with $\dot{\alpha}_p = \mathrm{d}\alpha_p / \mathrm{d}y_p$ and $\dot{\beta}_q = \mathrm{d}\beta_q / \mathrm{d}y_q$ we have

$$\begin{aligned}
[\mathrm{curl}\, V]_{ij} &= \sum_{p=1}^N \frac{\dot{\alpha}_p}{y_p} \left(w_j^p w_i^p - w_i^p w_j^p \right) + \sum_{q=1}^N \frac{\dot{\beta}_q}{y_q} \left(w_j^q [w_i^q]^\perp - w_i^q [w_j^q]^\perp \right) + \sum_{q=1}^d 2\beta_q \Lambda_{ij}^q \\
&= \sum_{q=1}^N \frac{\dot{\beta}_q}{y_q} \left(w_j^q [w_i^q]^\perp - w_i^q [w_j^q]^\perp \right) + \sum_{q=1}^d 2\beta_q \Lambda_{ij}^q.
\end{aligned} \tag{5.5}$$

  In particular, if $\beta_q \equiv 0$ $(1 \le q \le d)$ then $\mathrm{curl}\, V \equiv 0$. In fact choosing $\Phi_p = \Phi_p(y_p)$ so that $\mathrm{d}\Phi_p / \mathrm{d}y_p = y_p \alpha_p$ we have $V = \nabla \sum_p \Phi_p(y_p)$.
- If $\alpha_p(y) = \alpha_p(r)$, $\beta_q(y) = \beta_q(r)$ $(r = \|y\|)$ then with $\dot{\alpha}_p = \mathrm{d}\alpha_p / \mathrm{d}r$, $\dot{\beta}_q = \mathrm{d}\beta_q / \mathrm{d}r$ we have

$$
\begin{aligned}
[\mathrm{curl}\, V] &= \sum_{p=1}^{N}\sum_{k=1}^{N} \frac{\dot\alpha_p}{r}\left[w^p \otimes w^k - w^k \otimes w^p\right] \\
&\quad + \sum_{q=1}^{N}\sum_{k=1}^{N} \frac{\dot\beta_q}{r}\left([w^q]^\perp \otimes w^k - w^k \otimes [w^q]^\perp\right) + \sum_{q=1}^{d} 2\beta_q \Lambda^q.
\end{aligned}
\tag{5.6}
$$

If, additionally, $\alpha_p(y) = \alpha(r)$, $\beta_q(y) = \beta(r)$ for all $1 \le p, q \le N$ then we have

$$
\begin{aligned}
\mathrm{curl}\, V &= \frac{\dot\alpha}{r}\sum_{p=1}^{N}\sum_{k=1}^{N}\left[w^p \otimes w^k - w^k \otimes w^p\right] + \frac{\dot\beta}{r}\sum_{q=1}^{N}\sum_{k=1}^{N}\left([w^q]^\perp \otimes w^k - w^k \otimes [w^q]^\perp\right) \\
&\quad + \sum_{q=1}^{d} 2\beta\Lambda^q = \frac{\dot\beta}{r}\left(\sum_{q=1}^{N}[w^q]^\perp \otimes x - x \otimes \sum_{q=1}^{N}[w^q]^\perp\right) + \sum_{q=1}^{d} 2\beta\Lambda^q,
\end{aligned}
\tag{5.7}
$$

by virtue of $x = \sum_{p=1}^{N} w^p$ and thus $\sum_{p=1}^{N}\sum_{k=1}^{N}\left[w^p \otimes w^k - w^k \otimes w^p\right] \equiv 0$.

Let us now direct the above analysis towards the system $\Sigma[(u, \mathscr{P}), \Omega]$ and its whirl solutions. Indeed setting $V = \mathscr{L}[u] - \nabla F_\xi$ as in (5.1) and invoking Proposition 5 gives

$$
\begin{aligned}
\mathrm{curl}\left(\mathscr{L}[u] - \nabla F_\xi\right) &= \mathrm{curl} \sum_{p=1}^{N} \alpha_p w^p = \sum_{p=1}^{N}\sum_{k=1}^{N} \frac{\partial_k \alpha_p}{y_k}(y)\left[w^p \otimes w^k - w^k \otimes w^p\right] \\
&= \sum_{1 \le p < k \le N} \left(\frac{\partial_k \alpha_p}{y_k} - \frac{\partial_p \alpha_k}{y_p}\right)\left[w^p \otimes w^k - w^k \otimes w^p\right],
\end{aligned}
\tag{5.8}
$$

with $\alpha_p = -F_\xi |\nabla f_p|^2 - F_s$.[6] Now (5.8) $\equiv 0$ (recall the PDE $\mathscr{L}[u] = \nabla \mathscr{P}$) results in the vanishing of all the coefficients of $[w^p \otimes w^k - w^k \otimes w^p]$, i.e., $\partial_k \alpha_p / y_k - \partial_p \alpha_k / y_p \equiv 0$. We will analyse the implications of these conditions further in the following sections.

6 The complete solvability of $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ and $\Delta_H = \Delta_H(r)$

To get a better view of Theorem 2, the system $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ in (4.2) and the curl analysis in the previous section we take a closer look at the case $F(r, s, \xi) = H(r, s)\xi$ where $H = H(r, s)$ is a strictly positive function of class $\mathscr{C}^2$. The system (4.2) here is linear and has the decoupled form (with $1 \le i \le d$ and no summation over i):

[6] Note that for the sake of a uniform notation here we can regard f as being extended to an N-vector function when $n = 2d + 1$ by setting $f_N \equiv 0$.

$$
\mathbf{RS}[\mathsf{f}, \mathbb{A}_n] = \begin{cases} \operatorname{div}[\mathsf{A}_i(y)\nabla f_i] = 0 & \text{in } \mathbb{A}_n, \\ f_i = 0 & \text{on } (\partial\mathbb{A}_n)_a, \\ f_i = 2m_i\pi & \text{on } (\partial\mathbb{A}_n)_b, \\ \mathsf{A}_i(y)\partial_\nu f_i = 0 & \text{on } \Gamma_n, \end{cases} \tag{6.1}
$$

where $\mathsf{A}_i(y) = H(\|y\|, \|y\|^2)y_i^2\mathscr{J}(y)$ and as before $\mathscr{J}(y) = y_1 \ldots y_d$, $\mathsf{f} = (f_1, \ldots, f_d)$ and $\mathsf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$. The first claim is that this system has the unique solution $\mathsf{f} = \mathsf{f}(y, \mathsf{m})$ with $f_i(y) = 2m_i\pi\mathscr{H}(\|y\|)$ and the choice of profile curve $\mathscr{H} = \mathscr{H}(r) \in \mathscr{C}^2[a, b]$

$$
\mathscr{H}(r) = \frac{\mathsf{H}(r)}{\mathsf{H}(b)}, \qquad \mathsf{H}(r) = \int_a^r \frac{\mathrm{d}s}{s^{n+1}H(s, s^2)}, \qquad a \le r \le b. \tag{6.2}
$$

Whilst uniqueness is a consequence of Theorem 1, to show that f, as given, is a solution to $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ in (6.1), we proceed by first noting that the boundary conditions are satisfied in view of $\mathscr{H}(a) = 0$ and $\mathscr{H}(b) = 1$ along with $\mathsf{A}_i \equiv 0$ on Γ_n $(1 \le i \le d)$. Second, proceeding on to the PDE, direct differentiation and reference to (6.2) gives

$$
\frac{\partial f_i}{\partial y_\ell} = 2m_i\pi \frac{\dot{\mathsf{H}}(||y||)}{\mathsf{H}(b)} \frac{y_\ell}{||y||} = \frac{2m_i\pi y_\ell/\mathsf{H}(b)}{||y||^{n+2}H(||y||, ||y||^2)}, \qquad 1 \le \ell \le N. \tag{6.3}
$$

Now specialising first to even dimensions $n = 2d$, $N = d$, and writing $f = f_i$,

$$
\begin{aligned}
\operatorname{div}[\mathsf{A}_i(y)\nabla f] &= \operatorname{div}\Big[H(||y||, ||y||^2)y_i^2\mathscr{J}(y)\nabla f\Big] = \sum_{\ell=1}^{N} \frac{\partial}{\partial y_\ell}\left(\frac{2m_i\pi}{\mathsf{H}(b)} \frac{y_\ell \mathscr{J}(y)}{||y||^{n+2}} y_i^2\right) \\
&= \frac{2m_i\pi}{\mathsf{H}(b)} \sum_{\ell=1}^{d} \left(\frac{\mathscr{J}(y)y_i^2}{||y||^{n+2}} - (n+2)\frac{\mathscr{J}(y)y_i^2}{||y||^{n+4}} y_\ell^2 + 2\frac{y_\ell y_i \delta_{i\ell}}{||y||^{n+2}}\mathscr{J}(y) + \frac{\mathscr{J}(y)y_\ell}{||y||^{n+2}}\frac{y_i^2}{y_\ell}\right) \\
&= \frac{2m_i\pi}{\mathsf{H}(b)} \frac{\mathscr{J}(y)y_i}{||y||^{n+2}}\Big(dy_i - (2d+2)y_i + 2y_i + dy_i\Big) = 0.
\end{aligned} \tag{6.4}
$$

Next for $n = 2d + 1$, $N = d + 1$ we proceed similarly but recall that $y_N = x_n$. For the first $y_1, \ldots, y_d$ terms we have,

$$
\frac{2m_i\pi}{\mathsf{H}(b)} \sum_{\ell=1}^{d} \frac{\partial}{\partial y_\ell}\left(\frac{y_\ell\mathscr{J}(y)}{||y||^{n+2}} y_i^2\right) = \frac{2m_i\pi}{\mathsf{H}(b)} \frac{y_i\mathscr{J}(y)}{||y||^{n+2}}\left(dy_i - \frac{(2d+3)}{||y||^2} y_i \sum_{\ell=1}^{d} y_\ell^2 + 2y_i + dy_i\right).
$$

To this we add the Nth term in the divergence sum, which is then seen to be

$$
\frac{2m_i\pi}{\mathsf{H}(b)} \frac{\partial}{\partial y_N}\left(\frac{y_N\mathscr{J}(y)}{||y||^{n+2}} y_i^2\right) = \frac{2m_i\pi}{\mathsf{H}(b)} \frac{y_i\mathscr{J}(y)}{||y||^{n+2}}\left(y_i - \frac{(2d+3)}{||y||^2} y_i y_N^2\right).
$$

Combining this latter expression with the earlier sum above, therefore, gives

$$\begin{aligned}\operatorname{div}[\mathsf{A}_i(y)\nabla f] &= \operatorname{div}\Big[H(||y||, ||y||^2) y_i^2 \mathscr{J}(y)\nabla f\Big] = \frac{2m_i\pi}{\mathsf{H}(b)}\sum_{\ell=1}^{N}\frac{\partial}{\partial y_\ell}\left(\frac{y_\ell \mathscr{J}(y)}{||y||^{n+2}} y_i^2\right)\\ &= \frac{2m_i\pi}{\mathsf{H}(b)}\frac{y_i\mathscr{J}(y)}{||y||^{n+2}}\Big(dy_i - (2d+3)y_i + 2y_i + (d+1)y_i\Big) = 0.\end{aligned} \tag{6.5}$$

Hence the above in conjunction with Theorem 1 lead to the following result on the unique solvability of the restricted system (6.1).

Theorem 3 *Let $f_i = f_i(y, m_i) = 2m_i\pi\mathscr{H}(||y||)$ with $\mathscr{H} = \mathscr{H}(r)$ as in (6.2), $m_i \in \mathbb{Z}$ and $1 \le i \le d$. Then $\mathsf{f} = \mathsf{f}(y, \mathsf{m}) = (f_1, \ldots, f_d)$ in $\mathscr{C}^2(\overline{\mathbb{A}}_n, \mathbb{R}^d)$ is the unique solution to the system $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ in (6.1).*

Moving next on to the full system (1.2) and the PDE $\mathscr{L}[u] = \nabla\mathscr{P}$, let u denote the whirl associated with the matrix field $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ with f coming from Theorem 3. Recall that here the Euler–Lagrange operator $\mathscr{L}$ in (1.2) takes the explicit form

$$\begin{aligned}\mathscr{L}[u] &= [\nabla u]^t\Big\{\operatorname{div}\Big[H(|x|, |u|^2)\nabla u\Big] - H_s(|x|, |u|^2)|\nabla u|^2 u\Big\}\\ &= [\nabla u]^t[\nabla u]\nabla H(|x|, |u|^2) + H(|x|, |u|^2)[\nabla u]^t \Delta u - H_s(|x|, |u|^2)|\nabla u|^2 u.\end{aligned} \tag{6.6}$$

It then follows from Corollary 1 and upon utilising (4.7) that

$$\mathscr{L}[u] - \nabla H = -H\sum_{i=1}^{d}|\nabla f_i|^2 w^i - H_s\left(n + \sum_{j=1}^{d} y_j^2|\nabla f_j|^2\right)x, \tag{6.7}$$

where $H = H(r, r^2)$ and $H_s = H_s(r, r^2)$. A close inspection shows that the vector field on the right-hand side can be written in the form (5.2) with

$$\alpha_p(y) = -4m_p^2\pi^2 H\dot{\mathscr{H}}^2 - H_s\left(n + \sum_{j=1}^{d} 4m_j^2\pi^2 y_j^2\dot{\mathscr{H}}^2\right), \qquad 1 \le p \le N, \tag{6.8}$$

and $\beta_q \equiv 0$ for $1 \le q \le N$. (Note that when $n = 2d+1$ we extend m to an N-vector by setting $m_N = 0$.) Taking $\partial_k = \partial/\partial y_k$ $(1 \le k \le N)$ then leads to

$$\begin{aligned}\partial_k\alpha_p = -&\Bigg[4m_p^2\pi^2\left(\frac{\mathrm{d}H}{\mathrm{d}r}\dot{\mathscr{H}}^2 + 2H\dot{\mathscr{H}}\ddot{\mathscr{H}}\right)\frac{y_k}{||y||} + \frac{\mathrm{d}H_s}{\mathrm{d}r}\left(n + \sum_{j=1}^{d}4m_j^2\pi^2 y_j^2\dot{\mathscr{H}}^2\right)\frac{y_k}{||y||}\\ &+ 2H_s\dot{\mathscr{H}}^2\sum_{j=1}^{d}4m_j^2\pi^2 y_j\delta_{jk} + \left(2H_s\dot{\mathscr{H}}\ddot{\mathscr{H}}\sum_{j=1}^{d}4m_j^2\pi^2 y_j^2\right)\frac{y_k}{||y||}\Bigg].\end{aligned} \tag{6.9}$$

Now upon noting $\mathrm{d}H/\mathrm{d}r = H_r + 2rH_s$ and taking into account the relevant cancellations

$$
\begin{aligned}
&\frac{\partial_k \alpha_p}{y_k} - \frac{\partial_p \alpha_k}{y_p} \\
&= -\frac{1}{y_k}\Bigg[4m_p^2\pi^2\left(\frac{\mathrm{d}H}{\mathrm{d}r}\dot{\mathscr{H}}^2 + 2H\dot{\mathscr{H}}\ddot{\mathscr{H}}\right)\frac{y_k}{||y||} + \frac{\mathrm{d}H_s}{\mathrm{d}r}\left(n + \sum_{j=1}^{d} 4m_j^2\pi^2 y_j^2\dot{\mathscr{H}}^2\right)\frac{y_k}{||y||} \\
&\qquad + 2H_s\dot{\mathscr{H}}^2\sum_{j=1}^{d}4m_j^2\pi^2 y_j\delta_{jk} + \left(2H_s\dot{\mathscr{H}}\ddot{\mathscr{H}}\sum_{j=1}^{d}4m_j^2\pi^2 y_j^2\right)\frac{y_k}{||y||}\Bigg] \\
&+ \frac{1}{y_p}\Bigg[4m_k^2\pi^2\left(\frac{\mathrm{d}H}{\mathrm{d}r}\dot{\mathscr{H}}^2 + 2H\dot{\mathscr{H}}\ddot{\mathscr{H}}\right)\frac{y_p}{||y||} + \frac{\mathrm{d}H_s}{\mathrm{d}r}\left(n + \sum_{j=1}^{d} 4m_j^2\pi^2 y_j^2\dot{\mathscr{H}}^2\right)\frac{y_p}{||y||} \\
&\qquad + 2H_s\dot{\mathscr{H}}^2\sum_{j=1}^{d}4m_j^2\pi^2 y_j\delta_{jp} + \left(2H_s\dot{\mathscr{H}}\ddot{\mathscr{H}}\sum_{j=1}^{d}4m_j^2\pi^2 y_j^2\right)\frac{y_p}{||y||}\Bigg] \\
&= -4(m_p^2 - m_k^2)\pi^2\left[\frac{1}{r}(H_r + 2rH_s)\dot{\mathscr{H}}^2 + \frac{2}{r}H\dot{\mathscr{H}}\ddot{\mathscr{H}} - 2H_s\dot{\mathscr{H}}^2\right] \\
&= 4(m_p^2 - m_k^2)\pi^2\frac{\Delta_H\dot{\mathscr{H}}^2}{||y||^2},
\end{aligned}
\tag{6.10}
$$

where we have set $\Delta_H(r) = rH_r(r, r^2) + 2(n+1)H(r, r^2) + 4r^2H_s(r, r^2)$ and made use of the ODE $d/dr[r^{n+1}H(r, r^2)\dot{\mathscr{H}}] = 0$ [*cf.* (6.2)]. A reference to Proposition 5 now gives

$$
\begin{aligned}
\operatorname{curl}(\mathscr{L}[u] - \nabla H) &= \sum_{1\le p<k\le N}\left(\frac{\partial_k\alpha_p}{y_k} - \frac{\partial_p\alpha_k}{y_p}\right)\left[w^p\otimes w^k - w^k\otimes w^p\right] \\
&= \sum_{1\le p<k\le N} 4(m_p^2 - m_k^2)\pi^2\frac{\Delta_H\dot{\mathscr{H}}^2}{||y||^2}\left[w^p\otimes w^k - w^k\otimes w^p\right].
\end{aligned}
\tag{6.11}
$$

Theorem 4 *Let u be a whirl associated with the matrix field $\mathbf{Q}[\mathsf{f}] \in \mathscr{C}^2(\overline{\mathbb{A}}_n, \mathbf{SO}(n))$ satisfying $\mathbf{Q}[\mathsf{f}](y) = \mathbf{I}_n$ for $y \in (\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b$. Then u is a solution to the system $\Sigma[(u,\mathscr{P}), \mathbb{X}_n]$ with $\mathscr{L} = \mathscr{L}[u]$ as in (6.6) iff $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ is as described below.*

1. ($\Delta_H \not\equiv 0$ *on* $]a, b[$) *Here depending on the dimension n being even or odd we have*
   (i) *n even*: $\mathbf{Q}[\mathsf{f}](y) = \operatorname{diag}(\mathbf{R}[f_1(y)], \ldots, \mathbf{R}[f_d(y)])$ $(y \in \overline{\mathbb{A}}_n)$ *where*
   $$f_\ell(y) = 2m_\ell\pi\mathscr{H}(||y||), \quad 1 \le \ell \le d, \tag{6.12}$$
   with $m_1, \ldots, m_d \in \mathbb{Z}$ satisfying $|m_1| = \ldots = |m_d|$.
   (ii) *n odd*: $m_1 = \ldots = m_d = 0$ *and, therefore* $\mathbf{Q} \equiv \mathbf{I}_n$.

2. ($\varDelta_H \equiv 0$ *on* $]a, b[$) *Here* $\mathbf{Q}[\mathsf{f}](y) = \mathrm{diag}(\mathbf{R}[f_1(y)], \ldots, \mathbf{R}[f_d(y)])$ $(y \in \overline{\mathbb{A}}_n)$ *when* n *is even* $(n = 2d)$ *and* $\mathbf{Q}[\mathsf{f}](y) = \mathrm{diag}(\mathbf{R}[f_1(y)], \ldots, \mathbf{R}[f_d(y)], 1)$ $(y \in \overline{\mathbb{A}}_n)$ *when* n *is odd* $(n = 2d + 1)$. *In either case* f_ℓ *is as in* (6.12) *for each* $1 \le \ell \le d$ *with no restriction on the integers* $m_1, \ldots, m_d$.

Proof If u is a solution to $\Sigma[(u, \mathscr{P})]$ then the vector function $\mathsf{f} = (f_1, \ldots, f_d)$ is a solution to $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ and must, therefore, be precisely as described by Theorem 3. Now (6.11) and the fact that necessarily $\mathrm{curl}\,(\mathscr{L}[u] - \nabla H) \equiv 0$ gives $(m_p^2 - m_k^2)\varDelta_H \equiv 0$ for all $1 \le p < k \le N$. (Here the tensors $[w^p \otimes w^k - w^k \otimes w^p]$ are independent and vanish at most on the coordinate hyperplanes.) Thus if $\varDelta_H \not\equiv 0$, a continuity argument, and the above gives $|m_1| = \ldots = |m_N|$. When $n = 2d$ this is precisely as claimed and when $n = 2d + 1$ due to $m_N = 0$ this gives $m_1 = \ldots = m_N = 0$ and so $\mathbf{Q} \equiv \mathbf{I}_n$.

Conversely, if $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ is as described in part 1, then referring to (6.7), (6.8), it is easily seen that $\mathscr{L}[u] = \nabla \mathscr{P}$ and so u is a solution to $\Sigma[(u, \mathscr{P})]$. If $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ is as described in part 2, then noting $\varDelta_H(r) = rH_r(r, r^2) + 2(n+1)H(r, r^2) + 4r^2 H_s(r, r^2) \equiv 0$, it follows that the vector field $U = \mathscr{L}[u] - \nabla H + nH_s x$, can be written as a gradient, indeed, $U(x) = -\nabla[H\dot{\mathscr{H}}^2 \sum_{\ell=1}^d 2m_\ell^2 \pi^2 y_\ell^2]$, by virtue of [*see* (6.7)]

$$\mathscr{L}[u] - \nabla H + nH_s x = -H \sum_{\ell=1}^{d} |\nabla f_\ell|^2 w^\ell - H_s \sum_{\ell=1}^{d} y_\ell^2 |\nabla f_\ell|^2 x, \tag{6.13}$$

where f_ℓ $(1 \le \ell \le d)$ are as in (6.12). As a result again we have $\mathscr{L}[u] = \nabla \mathscr{P}$ and so it follows that u is a solution to $\Sigma[(u, \mathscr{P})]$.

7 Solvability of $\Sigma[(u, \mathscr{P}), \mathbb{X}_n]$ and an infinitude of extremisers for $\mathbb{I}[u, \mathbb{X}_n]$ in even dimensions

In this section, we restrict to even dimensions $n = 2d$ and prove the existence of an infinitude of whirl solutions to $\Sigma[(u, \mathscr{P}), \mathbb{X}_n]$ by taking $\mathbf{SO}(n)$-valued matrix fields $\mathbf{Q} = \mathbf{Q}[\mathsf{f}]$ whose vector functions $\mathsf{f} = (f_1, \ldots f_d)$ are suitable solutions to the reduced system $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$. In doing so the following naturally arising two-point boundary value problem plays a key role:

$$\mathbf{BVP}[\mathscr{G}, m] := \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}\left[r^{n+1} F_\xi\left(r, r^2, n + r^2\dot{\mathscr{G}}^2\right)\dot{\mathscr{G}}\right] = 0, & a < r < b, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m\pi. \end{cases} \tag{7.1}$$

In this regard, let us proceed with the following result.

Theorem 5 *Let* $n = 2d$ *and assume* $m_1 = \ldots = m_d =: m$. *Then* $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ *admits the unique solution* $\mathsf{f}(y, \mathsf{m}) = (f_1, \ldots, f_d)$ *where for each* $1 \le \ell \le d$, $f_\ell = f_\ell(y, m_\ell) = \mathscr{G}(||y||, m)$. *Here* $\mathscr{G} = \mathscr{G}(r) \in \mathscr{C}^2[a, b]$ *is the unique solution to* $\mathbf{BVP}$ *in* (7.1).

Proof First note that the boundary conditions on f are satisfied on $(\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b$ as a result of the imposed end-point conditions on $\mathscr{G}$ in **BVP**. Next proceeding on to the PDE in $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ we first observe that for any $1 \le \ell \le d$, $1 \le k \le N$ we have

$$\partial_k f_\ell = \partial_k \mathscr{G}(||y||) = \dot{\mathscr{G}} \frac{y_k}{r} \Rightarrow \sum_{j=1}^{d} y_j^2 |\nabla f_j|^2 = \sum_{j=1}^{d} y_j^2 \sum_{k=1}^{N} \dot{\mathscr{G}}^2 \frac{y_k^2}{r^2} = r^2 \dot{\mathscr{G}}^2. \tag{7.2}$$

Now as by assumption $n = 2d$ with $N = d$ it follows by direct calculation that, for any $1 \le i \le d$, and with $\mathscr{J}(y) = y_1 \ldots y_d$,

$$\begin{aligned} \operatorname{div}[\mathsf{A}_i(y, \nabla \mathsf{f}) \nabla f_i] &= \sum_{k=1}^{d} \partial_k \left[F_\xi(||y||, ||y||^2, n + \dot{\mathscr{G}}^2 ||y||^2) \frac{y_k}{||y||} y_i^2 \mathscr{J}(y) \dot{\mathscr{G}} \right] \\ &= \frac{y_i^2}{||y||} \mathscr{J}(y) \left\{ r \dot{\mathscr{G}} \frac{\mathrm{d}}{\mathrm{d}r} F_\xi + r F_\xi \ddot{\mathscr{G}} + (2d+1) F_\xi \dot{\mathscr{G}} \right\} \\ &= \frac{y_i^2 \mathscr{J}(y)}{||y||^{n+1}} \frac{\mathrm{d}}{\mathrm{d}r} \left[r^{n+1} F_\xi \left(r, r^2, n + r^2 \dot{\mathscr{G}}^2 \right) \dot{\mathscr{G}} \right], \end{aligned} \tag{7.3}$$

where for the sake of brevity in the second line we have written $F_\xi = F_\xi(r, r^2, n + r^2 \dot{\mathscr{G}}^2)$. This being so, under the assumption that $\mathscr{G}$ is a solution to **BVP**, we see that the above vanishes and so the PDE in $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$ is satisfied. The assertion is thus justified and the proof is complete.

Theorem 6 *For every $m \in \mathbb{Z}$ there exists a unique solution $\mathscr{G} \in \mathscr{C}^2[a, b]$ to the boundary value problem* (7.1).

Proof It is easily seen that $\mathbf{BVP}[\mathscr{G}, m]$ is the Euler–Lagrange equation associated with the energy functional

$$\mathbb{G}[\mathscr{G}; a, b] := \int_a^b F(r, r^2, n + r^2 \dot{\mathscr{G}}^2) r^{n-1} \, \mathrm{d}r, \tag{7.4}$$

over the Dirichlet space $\mathscr{B}^p_m[a, b] = \{\mathscr{G} \in W^{1,p}(a, b) : \mathscr{G}(a) = 0, \mathscr{G}(b) = 2m\pi\}$. The existence of a minimiser follows by an application of the direct methods of the calculus of variations (note that $p > 1$). The uniqueness of this minimiser follows from the uniform convexity of F in the third variable and a convexity argument as seen above. Finally the $\mathscr{C}^2$-regularity of the minimiser follows from the Tonelli–Hilbert–Weierstrass differentiability theorem (see [5] pp. 57–62).

Proceeding with the solution $\mathsf{f}(y, \mathsf{m}) = (f_1, \ldots, f_d)$ from Theorem 5, and noting that this is a solution to the system $\mathbf{RS}[\mathsf{f}, \mathbb{A}_n]$, we have upon invoking Corollary 1, that the corresponding differential action as a result reduces to $\mathscr{L}[u] = \nabla F_\xi - r[F_\xi \dot{\mathscr{G}}^2 + F_s]x$ with the arguments of F_ξ, F_s being $(r, s, \xi) = (r, r^2, n + r^2 \dot{\mathscr{G}}^2)$.

Theorem 7 *Assume $n \ge 2$ is even and for each $m \in \mathbb{Z}$ let $u = u(x, m)$ denote the whirl map associated with the matrix field $\mathbf{Q}[\mathsf{f}(y, \mathsf{m})]$ where $\mathsf{f}(y, \mathsf{m}) = (f_1, \ldots, f_d)$ is the map from Theorem 5. Specifically, $u(x, m) = \mathbf{Q}[\mathsf{f}(y, \mathsf{m})]x = \exp\{\mathscr{G}(r; m)\mathbf{H}\}x$*

with $\mathbf{H} = \mathrm{diag}(\mathbf{J}, \ldots, \mathbf{J})$ *and* $\mathscr{G} \in \mathscr{C}^2[a, b]$ *the unique solution to the boundary value problem in* (7.1). *Then* u *is a solution to* $\Sigma[(u, \mathscr{P}), \mathbb{X}_n]$. *In particular* $\mathscr{L}[u] = \nabla\mathscr{P}$ *where the hydrostatic pressure up to an additive constant is given by*

$$\mathscr{P} = \mathscr{P}(x) = F_\xi(|x|, |x|^2, n + |x|^2\dot{\mathscr{G}}^2(|x|)) + G(|x|). \tag{7.5}$$

Here $G = G(r)$ *satisfies* $\nabla G = -[F_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}^2 + F_s(r, r^2, n + r^2\dot{\mathscr{G}}^2)]x$. *As a result the system* $\Sigma[(u, \mathscr{P}), \mathbb{X}_n]$ *has an infinitude of whirl solutions of class* $\mathscr{C}^2$.

Proof As seen earlier in Sect. 2 we have $\det \nabla u \equiv 1$ whilst for the boundary conditions, we have $\mathscr{G}(a; m) = 0$ leading to $u(x, m) = \exp\{\mathscr{G}(a)\mathbf{H}\}x = x$ for $|x| = a$ and $\mathscr{G}(b; m) = 2m\pi$, leading to $u(x, m) = \exp\{2m\pi\mathrm{diag}(\mathbf{J}, \ldots, \mathbf{J})\}x = \mathbf{I}_n x = x$ for $|x| = b$. It, therefore, remains to justify $\mathscr{L}[u] = \nabla\mathscr{P}$ and for that it suffices to write

$$\begin{aligned}\mathscr{L}[u = \mathbf{Q}[\mathsf{f}(y, \mathsf{m})]x] &= [\nabla u]^t\Big\{\mathrm{div}\Big[F_\xi(|x|, |u|^2, |\nabla u|^2)\nabla u\Big] - F_s(|x|, |u|^2, |\nabla u|^2)u\Big\} \\ &= \nabla F_\xi - \sum_{\ell=1}^{N} F_\xi \partial_\ell \mathbf{Q}^t \partial_\ell \mathbf{Q} x - F_s x = \nabla F_\xi - \sum_{i=1}^{N}\Big[F_\xi|\nabla f_i|^2 + F_s\Big]w^i \\ &= \nabla F_\xi(|x|, |x|^2, n + |x|^2\dot{\mathscr{G}}^2(|x|)) + \nabla G(|x|),\end{aligned} \tag{7.6}$$

where G is a primitive of $g(r) = -rF_\xi(r, r^2, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}^2 - rF_s(r, r^2, n + r^2\dot{\mathscr{G}}^2)$. This, therefore, completes the proof.

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